Dinesh Shenoy · Roberto Rosas

Problems & Solutions in Inventory Management

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Dinesh Shenoy
Tecnológico de Monterrey
Campus León, Mexico

Roberto Rosas
Tecnológico de Monterrey
Campus León, Mexico

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Preface

When we started writing the manuscript for this book, we had three principles in mind:

– Focus on the academic audience.
– Use simple language.
– Match the theoretical content with numerical examples.

We are glad we have been able to stick to these principles.

This book has been written for undergraduate business as well as industrial engineering students who are taking a course on inventory management or operations management. We believe the content presented in this book is just enough and has been organized well to keep the reader engaged. We have worked out several numerical problems in inventory management. This would particularly come in handy for instructors teaching a course on inventory management.

This book has been divided into four parts with the first part dealing with basic inventory management concepts and terms, including relevant inventory costs and methods of computing those. Toward the end of Chap. 2, we present a case study – Rosettas Tortilleria. This case study has been treated as a running example. Different scenarios for the same case have been presented throughout the book to give readers a common ground for learning. We believe the reader would be able to relate to reality and learn the concepts faster.

The second part of the book has four chapters. In these chapters, we discuss single-item inventory models including items with deterministic demand (without and with shortages allowed), dynamic demand (quantity discount), time-varying demand, and stochastic inventory models. More than 40 solved problems and 4 case studies have been presented in this part of the book.

The third part of the book deals with inventory models involving multiple items. This part of the book has two chapters in which we discuss inventory models subject to constraints (budget, space, and number of orders) as well as selective inventory control techniques. Coordinated replenishment of items is also covered in this part of the book. Solved numerical problems and case studies have also been included in this part.

The fourth and final part of the book discusses advanced inventory models including models for perishable and style items, maintenance and repairable inventory, and two-stage, multi-echelon inventory models. This may not be part of an undergraduate course curriculum; however, those with a keen mind on inventory management would find this section of the book very interesting.

Campus León, Mexico

Dinesh Shenoy
Roberto Rosas

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Contents

Part I
Basics of Inventory Management

Chapter 1
Introduction to Inventory Management

1.1 What Is Inventory?

APICS[1] *Dictionary* (2015) defines inventory as

> Those stocks or items used to support production (raw materials and work-in-process items), supporting activities (maintenance, repair and operating supplies), and customer service (finished goods and spare parts).

Items that are used in production include the following:

- Raw materials: These are primary ingredients that are used in making of a product. For a firm manufacturing automobile tires, for example, natural rubber would be the raw material. Leather (or fabric) and plastic would be primary raw materials for a firm manufacturing shoes. Manufacturers procure raw materials and transform those into finished goods using their production processes.
- Work-in-process: These are the stock of materials on which the production processes have started but have not been completed. These materials are not yet ready for sale.

Items that are used in supporting manufacturing activities are referred to as Maintenance, Repair and Operating (MRO) supplies. Transformation of materials from their raw form into one that is saleable requires production equipment. Production equipment are characterized by failures. These equipment need maintenance activities to be performed that maximize their availability. Performing maintenance activities on production equipment requires resources – for example, a mechanic may detect an oil leak while performing a simple cleaning operation on an equipment. Fixing the oil leak may require new oil seals. These oil seals, while not used directly in the production processes, are absolutely required by the production processes. It is important that the manufacturing firm also stocks oil

[1]American Production Inventory Control Society, established in 1957, is now called APICS.

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seals in their inventory system. Note that these items are supporting the manufacturing processes and are not sold by the firm stocking it. In this book, we refer to these items as maintenance items or maintenance materials.

As per *APICS Dictionary*, items used in customer service include finished goods and spare parts. From a manufacturing perspective, finished goods are items that have been completely transformed, and are now ready for sale. Tires, shoes, etc. are examples of finished goods. Firms also manufacture parts of equipment that are sold separately. These are called spare parts.

Organizations treat and classify inventory differently depending on whether they are merchandisers (i.e., firms that only trade ready-to-sell materials) or manufacturers (i.e., firms that transform materials). Manufacturers usually categorize their inventory into raw materials, work-in-process, and finished goods. On the other hand, merchandisers buy materials that are ready for sale and sell it to customers. Merchandisers, therefore, have just one classification for inventory.

Whether a firm is manufacturing products, providing services, or just trading, all operations of the firm need inventory. Managers in charge of inventory are expected to ensure the stock of items required by their firm is available at the right time and in the required quantity. Table 1.1 illustrates the types of inventory used by different sectors of the industry.

Notice that while cleaning liquid is a supporting material for the pizza outlet, the same is considered to be a raw material for a firm engaged in cleaning services.

Table 1.1 Examples of inventory

Entity	Type	Raw materials	Work-in-process	Finished goods	Other supporting items
Pizza outlet	Manufacturing	Flour, oil, tomato sauce, cheese, pineapple, items that make up the toppings, pizza delivery boxes, etc.	Dough, pizzas in the oven, pizza toppings	Packed pizzas ready for delivery, bottled drinks, and colas	Cooking gas (for the oven), cleaning liquid
Blood bank	Service	Whole blood units	Blood units being processed for separation	Red blood cells, plasma, platelets	Nutritive solutions required for storage of blood components
Vegetable vendor	Trading	Fruits and vegetables			Packaging materials
Cleaning services	Service	Cleaning liquid, detergents, soaps, acids, etc.			Mops, vacuum cleaners.

1.2 Functions of Inventory

Firms maintain inventory because of economic reasons. Carrying inventory makes it more economical to produce their products than by not carrying it (Harding and Harding 2001). Also, inventory carried by firms have some purpose (or function), of which following are some important ones (see Mahadevan 2015; Hill and Hill 2012):

- Decoupling
- Cycle (or Cyclical)
- Pipeline
- Buffer

These functions are described briefly in the following section.

1.2.1 Decoupling Inventory

Decoupling inventory is one that detaches a manufacturing process from another. Manufacturing a product typically requires several workstations, usually in a sequence. Raw material is fed to the first workstation, and after processing it is sent to the second workstation, and so on, until the last workstation in the sequence finishes the product. Workstations have different processing capacities – some may finish processing faster while others may take more time. A workstation in the production line may also encounter a failure. If this happens, all upstream workstations have to wait until the failed workstation is back up online. Decoupling inventory is an intermediary inventory maintained between two workstations. The function of this inventory is to help smoothen the workflow between workstations and minimize the impact of fluctuations.

1.2.2 Cycle Inventory

An organization does not procure all input materials at one go. In most cases, they also do not order one unit at a time. They order in batches, usually in lot sizes that are larger than those demanded by customers (see Chopra and Meindl 2010, pp. 246). For example, a manufacturing firm requiring 50,000 tons annually may order 4000 tons of steel to start with. The next order would be placed when the on-hand inventory is nearly depleted. Cycle inventory is the result of ordering materials in batches. Batch ordering is preferred by firms because they derive benefits of economies of scale.

Several economic factors go into the determination of optimal batch size which will be dealt with in the following chapters. Mathematically, if Q is the size of the order, then CI, the average cycle inventory, is given by

$$\mathrm{CI} = \frac{Q}{2} \tag{1.1}$$

1.2.3 Pipeline Inventory

Manufacturing organizations procure input materials from their suppliers. The ordered materials are not received instantaneously. There is a delay between placement of orders and receipt of materials due to geographical distance between the supplier and the manufacturer. Inventory that is in transit (in trucks, ships, etc.) is referred to as pipeline inventory. Function of this inventory it to account for uncertainties in supply lead time. Mathematically,

$$\mathrm{PI} = DL \tag{1.2}$$

where PI is the pipeline inventory, D is the demand, and L is the lead time. It should be noted that the units for demand and lead time must be the same. That is, if demand is expressed in units per day, the lead time must also be in days.

Solved Problem 1.1
At the beginning of every month, a bicycle manufacturer, based in Monterrey, Mexico, places an order to procure lightweight steel tubes from their suppliers based in Leon, Mexico. Their average requirement is 1000 tons per week, and the procurement lead time is 14 days. Determine cycle and pipeline inventory. Assume 4 weeks in a month.

Solution
Average weekly demand for steel tubes is 1000 tons. The monthly demand (assuming 4 weeks in a month) is 4000 tons. The cycle inventory from Eq. 1.1 is

$$\mathrm{CI} = \frac{Q}{2} = \frac{4000}{2} = 2000 \text{ units}$$

Since the procurement lead time is 14 days (or 2 weeks), using Eq. 1.2, we get pipeline inventory. Substituting the values, we get

$$\mathrm{PI} = DL = 1000 \times 2 = 2000 \text{ tons}$$

1.2.4 Buffer Inventory

Buffer inventory, or safety stock, is maintained by firms to counter uncertainty in demand. Buffer inventory reduces the probability of running out of stock.

1.3 Inventory Management: Key Issues

As discussed earlier, inventory managers are expected to ensure the stock of items required by their firm is available at the right time, and in the required quantity. With this perspective, management of inventory in any firm involves answering the following three questions (Anderson et al. 2016; Gaither 1987; Silver et al. 1998):

(a) How frequently should the inventory status for an item be reviewed?
(b) When should a replenishment order be placed?
(c) What should be the order size?

While at the outset, answering these questions may look to be very simple; however, the characteristics of items being managed make inventory management complex and challenging. Following are some of the characteristics of inventory items.

- *Demand*: This is a key characteristic of an item in any inventory system. Demand is generated by people outside the firm, and therefore is mostly uncontrollable (see Naddor 1966). Nevertheless, the pattern of demand may be studied to address the inventory management problem of the firm. Table 1.2 shows a list of terms that may be used to describe demand.
- *Replenishment Lead Time*: Lead time is the time between placement of a replenishment order and the actual receipt of stock. Just like in the case of demand, lead times for an item may be constant, varying, or may be known with a probability distribution. Theoretically, in some cases, the lead times may be very small and insignificant. This case may be referred to as instantaneous replenishment.
- *Inventory Level and Review Times*: The inventory level of an item may be known at all times. In some cases, it may not be possible (or may not be cost-effective) to monitor the level of an item at all times. Instead, the level of items is reviewed at certain predetermined times.
- *Lifetime & Reparability*: An item may not always have an indefinite lifetime. Its utility value may drop to zero at some point after which it may not have any takers (Nahmias 2005). Also, an item may fail but may be repairable.

Characteristics of an item would be different from one another, and so would be their inventory decisions.

Table 1.2 Terms used to describe demand

Term	Meaning
Deterministic	The size of the demand for an item is known precisely, in advance
Constant	The size of the demand for an item remains the same from one period to another
Varying	The size of the demand is not constant and keeps changing between periods
Stochastic/ probabilistic	Demand is not known, but based on historical data we may be able to fit it to a probability distribution

1.4 Inventory Management: An Overview of Mathematical Models

An inventory model must essentially address the three main issues mentioned in the previous section, keeping in view characteristics of items being managed. In this section, we present a classification of mathematical models that have been developed to address these problems. Several factors have been used by researchers to classify inventory models Vrat (2014). Some of these factors include:

- Timing of inventory status review (continuous, periodic)
- Nature of demand of item (deterministic, probabilistic, varying by period, etc.)
- Nature of replenishment lead time (constant, varying, etc.)
- Number of items under management (single, multiple)
- Number of locations or supply sources (single, multi-echelon)
- Possibility of repetitive orders (single period, multiple periods)
- Lifetime of items being managed (perishable, infinite lifetime)
- Other factors (reparability of items, constraints, quantity discounts, coordinated replenishment, etc.)

Figure 1.1 shows a classification of inventory models. This classification is by no means exhaustive, but it does give the reader an idea of the breadth of inventory management solutions. This classification has been used to structure the contents of this book.

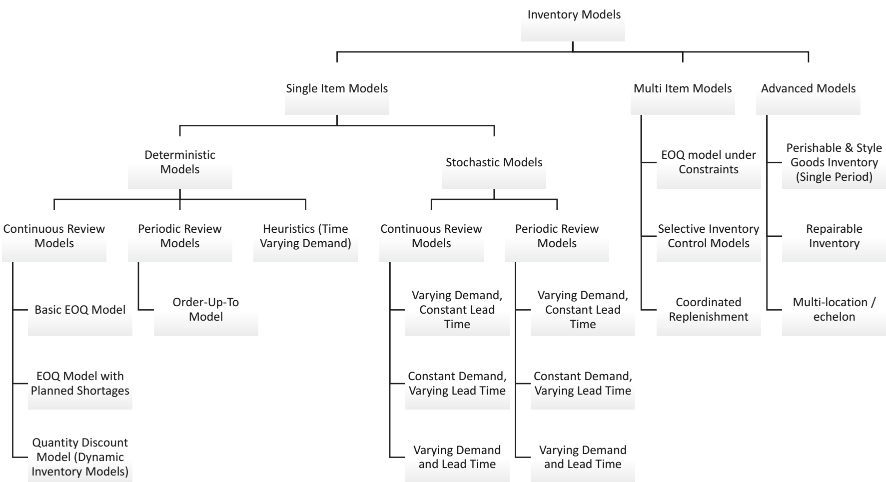

Fig. 1.1 Classification of inventory models

1.5 Summary

In this chapter, we discussed the meaning of the term inventory and described the different types of inventories – raw materials, work-in-process, finished goods, and maintenance items – used by businesses. There are several reasons why businesses hold inventories. Decoupling inventories are held to detach a process from another while cycle and pipeline inventories are held for minimizing inventory costs. Buffer or safety stocks are held to minimize uncertainty in demand.

We also learned in this chapter that inventory management revolves around finding answers to three most important questions:

(a) How frequently should the inventory status for an item be reviewed?
(b) When should a replenishment order be placed? and
(c) What should be the order size?

Inventory managers need to consider the characteristics of items – such as demand, replenishment lead time, the timing of review, and item lifetime – being managed while answering these questions.

We also presented a classification of inventory models that gives the reader an idea of the breadth of inventory management solutions.

1.6 Case Study: Tequila Production Process

Julius Tequila is a world famous distillery located in the state of Guanajuato, Mexico. The distillery makes the Julius brand of premium tequila. The distillery procures tons of full-grown agave piñas[2] from the farms all over Mexico. An agave plant that has grown for about 8 years to the size of a soccer ball is considered to be good for producing tequila. The piñas are first cut into several uniform pieces and washed with clean water. The cut pieces of piñas, the insides of which are white in color, are then fed into an autoclave, a chamber that functions like a pressure cooker. The pieces of raw piña are cooked for about 72 h in the autoclave. At the end of the cooking process, the color of the piña turns golden brown.

The cooked golden brown piña pieces are then conveyed to a crusher where a screw-crushing facility, consisting of several parallel rotating screw crushers, is employed to continuously crush and extract the piña juice. The juice extracted from the piña is separated from its fiber using a simple, sieve-based filtering process. The distilled juice is then transferred to another chamber where it is mixed with yeast. The mixture of juice and yeast is allowed to ferment and settle. This settling down process takes a long time, and it does so in three layers. The top (the head) and the bottom layers (the tail) of the settled partially fermented liquid is used in making

[2]Fruit or the bulb found at the base of the Agave plant.

lower grades of tequila while the middle one is used in making the premium version.

The different versions of the partially fermented liquid are then transferred to wooden barrels that are stored in dark, cool storage chambers for several hours (minimum 20,000).

After more than 2 years of storage, the liquid is taken out of the barrels and transferred into blue-hued bottles (capacity of 2 liters each). These bottles with tequila are now ready for commercial sale. These bottles are ferried to distributor warehouses around the world.

Mild, food-quality compliant detergents are used to clean the continuous process machines before the next load of agave piñas are loaded into the system.

Case Study Questions
From the above case, classify the inventory in tequila production into

(a) raw materials
(b) work-in-process, and
(c) finished goods inventory

1.7 Practice Problems

Problem 1.1
A shoe manufacturer has a daily requirement of 100 shoe soles. The manufacturer places an order for a batch size of 1000 shoe soles from a supplier. Once an order is placed, it takes 3 days to reach the manufacturing location. Determine cycle and pipeline inventory.

Answer
Cycle inventory is 500 shoe soles. Pipeline inventory is 300 soles.

Problem 1.2
Review different textbooks on Inventory Management from your library. Compile a list of definitions for the term "inventory" used by the authors of those books.

Problem 1.3
Consider a shoe manufacturing facility. Identify all types of inventory used in manufacturing one model of a fast-moving shoe.

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Chapter 2
Inventory Control Systems: Design Factors

2.1 Design Factors

One of the takeaways from the previous chapter is that the core of the inventory management problem is to address the following issues:

(a) How frequently should the inventory status for an item be reviewed?
(b) When should a replenishment order be placed?
(c) What should be the order size?

An inventory control system is one that integrates these three factors into one whole. Frequency, timing, and order size are the key design factors of any inventory control system. Each of these design factors is discussed in detail in this section.

2.1.1 Review Frequency

Depending on the nature of their business, a firm may decide to review the status of inventory either continuously or at discrete, but fixed, points in time. A continuous review inventory system is one where the inventory status of an item (or items) is known at all times (Silver et al. 1998; Nahmias 2005). An example of this would be a retail shop using a Point-of-Sale (POS) system to record incoming and outgoing materials. Every time a sale is made, the POS system captures the information of the product sold and updates the stock level of the product. This way the owner of the retail shop has real-time information on the stock levels of products and can make appropriate inventory decisions.

A periodic review system is one where the stock level is reviewed every T units of time. In other words, if the first review is made at time 0, the next review is made at time T, the third review at time $2\,T$, and so on. An example of this system would be a pharmacy that sells drugs and medicines to general public. Every Monday, a

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representative of a drug distributor arrives at a pharmacy to sell drugs and medicines. The pharmacist quickly reviews the stock of the drugs on hand and makes a purchase decision. In firms that use this system, the stock levels are not known between review points. No action is taken if an item runs out of stock between review periods.

2.1.2 Timing of Replenishment Order

The timing of an order is another key concern for inventory managers. Ideally, the timing must be such that the ordered items are received just as the last item on hand gets consumed (or sold). With regard to timing of placement of an order, two options are available:

- Place an order the moment the current inventory level reaches a predetermined level.
- Place an order at predetermined review points

While the first option seems to be feasible for continuous review systems, the second one would fit a periodic review system.

2.1.3 Size of Replenishment Order

The size of a replenishment order depends on several factors such as the ordering cost, the holding cost, the shortage cost, etc. If we order less, then we would need to order more frequently. This may increase ordering costs. There also exists the risk of running out of stock and the inability to meet customer demand. A bigger order size may result in higher carrying costs and the inability to store all received materials due to limited space. A good inventory system would be one where the order size is just right and takes into account all these factors. Again, two options are most popular:

- Determine and place an order for a fixed quantity that minimizes the inventory costs.
- Compute an order quantity such that it takes the stock level back to predetermine maximum level.

A continuous review inventory system can be designed considering both options. A periodic review would be a good fit for the second option.

2.2 Review of Inventory Control Systems

As discussed earlier, an inventory control system is one that integrates the three design factors into one whole. While adopting an inventory control system,[1] a firm must keep in view the design factors discussed in the previous section and address those appropriately. Several combinations and variants of inventory control systems are available in literature. The following are among the popular ones:

- Continuous Review, Fixed Order Quantity (s, Q) System.
- Continuous Review, Order-Up-to-Level (s, S) System.
- Periodic Review, Order-Up-to-Level (T, S) System.

In this section, we discuss each of the above in detail.

2.2.1 Continuous Review, Fixed Order Quantity (s, Q) *System*

In a firm that uses a continuous review system, the inventory position of an item is monitored continuously and is known at all times. Inventory position of an item is defined as the number of items held currently in stock plus the number of items on order. As demand arises, items are withdrawn from inventory. Simultaneously, the inventory position is updated. This process continues until the inventory level reaches a predetermined level, s, referred to as the reorder point. At this point, a new replenishment order of size Q is placed, which is filled after time L, referred to as the lead time. Receipt of the order increases the inventory position. The process of order-point, order-quantity system is illustrated in Fig. 2.1.

This system is also known as the (s, Q) system and the two-bin system.[2] Firms that adopt a continuous review system must decide on the following with respect to managing their inventory:

- An optimal fixed order size, Q.
- A reorder level, s.

Chapters 3 and 6 of this book deal with computation of these decision variables as well as the assumptions involved. Supermarkets and large retail stores are examples of organizations that would benefit from using this type of inventory system.

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[1] Few authors use the term Inventory Control Policy.

[2] In real-life implementation, an item is stored in two bins. Items are withdrawn from the first bin as demand arises. The moment the first bin is empty an order is placed. The second bin acts as the reorder point.

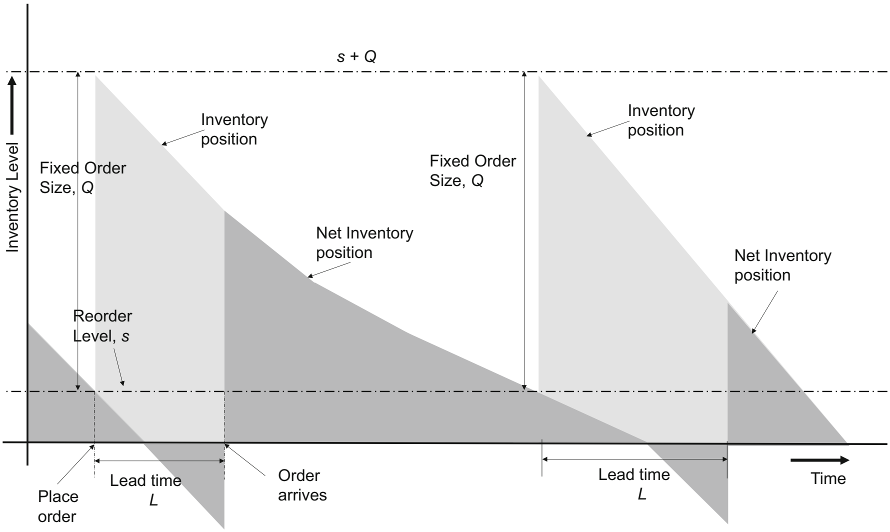

Fig. 2.1 Continuous review, fixed order quantity system

2.2.2 *Continuous Review, Order-Up-to-Level* (s, S) *System*

This continuous review-based inventory control system is similar to the order-point, order-level system with one difference – the order quantity in this system is variable. This system is also referred to as the (s, S) system or the Min–Max system.[3] When the inventory level reaches a predetermined level, s, referred to as the reorder point, a replenishment order of size Q is placed so as to raise the inventory level to the maximum level, S. Please note the order size Q in this system is variable. The decision variables in this system are as follows:

- A reorder level, s.
- The maximum inventory level, S.
- A variable order size, Q, that would bring the inventory level back up to S.

The process of order-point, order-up-to-level system is illustrated in Fig. 2.2.

2.2.3 *Periodic Review, Order-Up-to-Level* (T, S) *System*

In this periodic review-based inventory control system, also called the (T, S) system, inventory level of items is reviewed at predetermined, fixed points in time. If the first status review happens at time T, the second review would be

[3]This is because the inventory level is usually between a minimum (s) and maximum (S) levels.

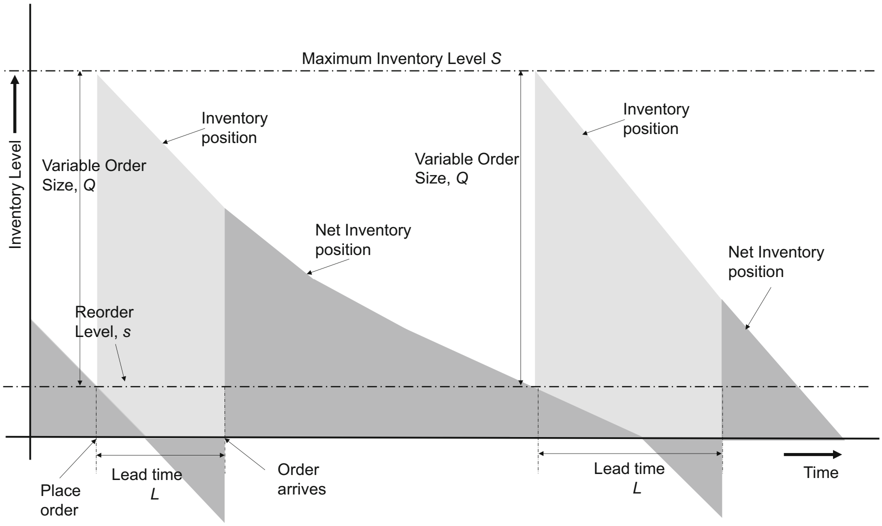

Fig. 2.2 Continuous review, order-up-to-level system

carried out at time $2T$, and so on. Following are the decision variables in organizations that use this system:

- An optimal fixed time, T, between replenishment orders.
- The maximum inventory level, S.
- A variable order size, Q. The order size is computed at the time of review. Depending on the inventory level (amount of inventory of hand and on order) at the time of review, the order size is computed that would bring the inventory level back up to S.

When the duration between review periods is long, there is a possibility of inventory position falling to low levels. In some cases, stockout situations may also be encountered (Fig. 2.3).

Table 2.1 shows a comparison of these three inventory control systems.

Other variants of inventory systems do exist in practice, but the ones described above are among the popular ones. Chapters 3 and 6 of this book describe the determination of the decision variables such as the reorder level, s, the order quantity, Q, the maximum permissible inventory level S, and the optimal time between review points, T.

2.3 Inventory Costs

An inventory control system that minimizes the total inventory costs would be the one an organization needs to adopt (Anderson et al. 2016; Chopra and Meindl 2010; Gaither 1987). In this section, we discuss the relevant costs involved in inventory management and the method to compute those with a common example.

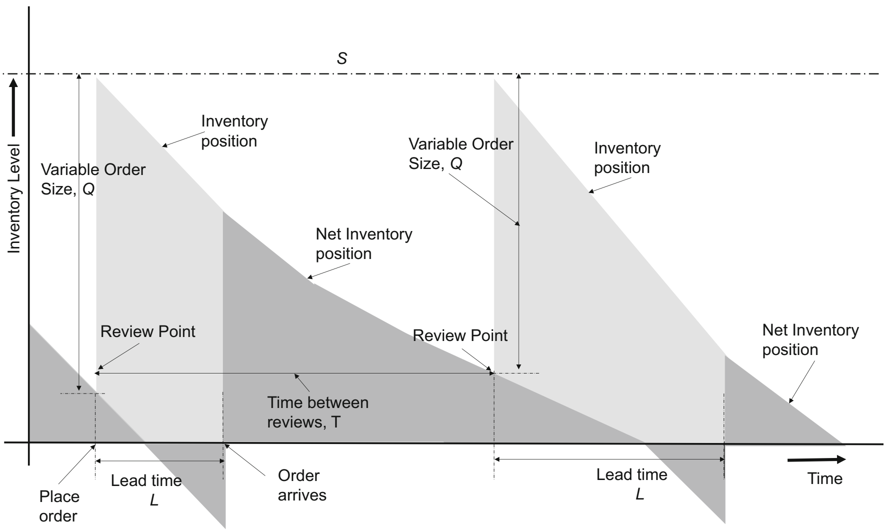

Fig. 2.3 Periodic review, order-up-to-level system

Table 2.1 Comparison of inventory control systems

Factor	Continuous review, fixed order quantity system	Continuous review, variable order quantity system	Periodic review, variable order quantity system
Monitoring of inventory level	Continuous	Continuous	Discrete points in time
Order size, *Q*	Fixed	Variable	Variable
Reorder level, *s*	Fixed	Fixed	Variable
Trigger for order	Event-based, when inventory level falls to s	Event-based, when inventory level falls to s	Time-based, at the review point, T
Time between stock reviews, *T*	Variable	Variable	Fixed
Size of safety stock, SS	Medium	Medium	High

Consider a vegetable vendor that procures fresh vegetables each day from a wholesale market and sells it to retail customers. The vegetable vendor visits the wholesale market early in the morning to procure large quantities of fresh vegetables and fruits. He invests some money to procure the same, and hopes to sell those to individuals and households later. Once the items are procured, the vendor transports these items to his retail outlet. On reaching the retail outlet, his assistant helps him unload and arrange some of them neatly in pallets in the front desk where customers can see, inspect, and purchase. Remaining items are taken to the cold storage at the back of the outlet where they would be stored temporarily. Throughout the day, customers arrive at the retail outlet to purchase vegetables and fruits of their choice. As the stock of items in the front desk gets lower, the vendor's assistant replenishes those pallets with items stored in the cold storage. The vendor and his assistant service customer requests in exchange for money. Toward evening, one of the following events may happen:

- All items that were procured in the morning have been sold, and no more customer requests may be serviced.
- A few items are still unsold, but since it is late evening no more customers may be expected. These items, if good for sale may continue to remain in the cold storage overnight and would be made available for sale the next morning. Items that are not good for sale are discarded.

Inventory costs are generally categorized into three types – carrying costs, ordering costs, and shortage costs. Methods of computing these types of costs with reference to the vegetable vendor case are described in detail in the following subsections.

2.3.1 Carrying Costs

Inventory carrying cost[4] is a cost associated with temporary storage of an item until it is sold. The carrying costs, C_h, expressed in terms of $ per unit per year, is given by

$$C_h = iC \tag{2.1}$$

where

C_h is the carrying (or holding) cost; i is the inventory carrying rate, expressed in % per year, and C is the unit price (cost) of the item, also expressed in $ per unit.

Assuming an inventory carrying rate of 25% per year, an item that has a cost of $100 would have a carrying cost of $C_h = 0.25 \times 100 = \$25$ per year (or $75 if it is held in inventory for 3 years). In this case, we assumed an inventory carrying rate. Let us now discuss the method to arrive at the inventory carrying rate, i. With reference to the vegetable vendor scenario, the inventory carrying rate, i, is an aggregation of the following cost components:

- Cost of capital: The vegetable vendor initially invests a large amount of money in procuring items that are meant for sale in future. Instead of investing money in procuring these items, the vendor could invest the money in an alternative proposal and reap profits. This is referred to as opportunity cost and also as the cost of capital. This cost is usually expressed in terms of percentage.
- Cost of storage: It would cost the vegetable vendor to provide physical space to store the procured items. This would include components such as the rent he has

[4] Some researchers refer to this cost also as holding costs or storage costs. We use the term holding cost and carrying cost interchangeably in this book.

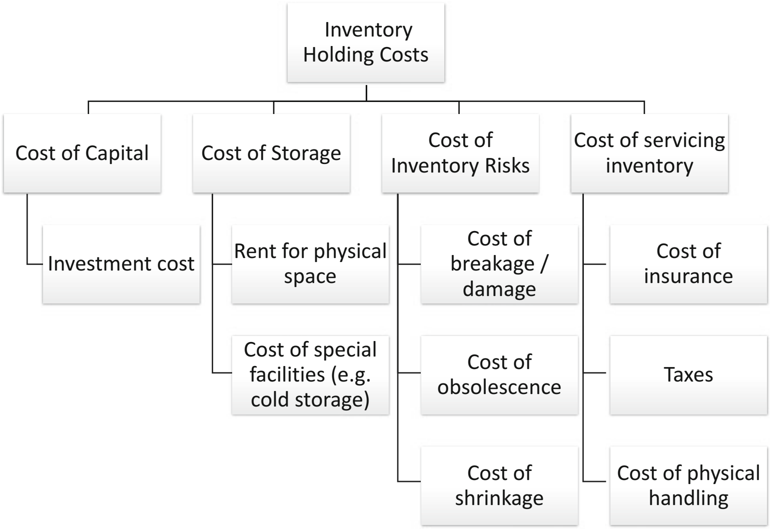

Fig. 2.4 Breakup of inventory carrying costs

Table 2.2 Data for Solved Problem 2.1

Type of cost	Values (per year)
Cost of capital	8.50%
Cost due to breakage	6.50%
Rent paid toward physical space for storage	3.50%
Premium paid to insure inventory	0.25%
Tax	1.00%

to pay for the retail outlet and the cost of making available the cold storage facility.

- Cost of inventory risk: Some items may get damaged during transportation. Some items may deteriorate during the day. These items may not be fit for sale and will have to be disposed off.
- Cost of servicing inventory: This includes taxes that the vendor needs to pay as well as the cost of insuring the items. It also includes the cost of people (wages) as well as maintenance of vehicles (such as forklift trucks), if any, involved in physical handling of materials inside the storage area/warehouse.

Figure 2.4 shows the breakup of the inventory carrying costs.

Solved Problem 2.1
Table 2.2 shows inventory data collated for one item. Compute the following:

(a) Inventory carrying rate
(b) The *annual carrying cost* of an item that costs $20

(c) The *total annual carrying cost* for 15 items that cost \$20 per unit
(d) The *total carrying cost* for 15 items that cost \$20 per unit and are held in inventory for a period of 2 years

Solution

(a) Refer to discussions in Sect. 2.3.1. The carrying rate is an aggregation of

- Cost of capital (8.50%);
- Cost of storage (3.50%);
- Cost of inventory risk (6.50%);
- Cost of inventory servicing (0.25% + 1.00%).

Therefore, the carrying rate is

$$= 8.50 + 3.50 + 6.50 + 0.25 + 1.00 = 19.75\%\text{per year.}$$

(b) Using Eq. 2.1, we can compute the annual carrying cost of an item that costs \$20 per unit, which is

$$C_h = \frac{19.75}{100} \times 20 = \$3.95.$$

The carrying cost is \$3.95 per unit per year.

(c) The total annual carrying cost of 15 items that cost \$20 per unit is

$$C_h = \frac{19.75}{100} \times 20 \times 15 = \$59.25.$$

The total carrying cost is \$59.25 per year.

(d) The total carrying cost for 15 items that cost \$20 per unit held in inventory over a 2-year period would be

$$C_h = \frac{19.75}{100} \times 20 \times 15 \times 2 = \$118.50.$$

The total carrying cost over a 2-year period is \$118.50.

2.3.2 Ordering Costs

Ordering cost is the cost associated with placing an order for an item and receiving it into the inventory system. It consists of the following components (Mahadevan 2015):

- Cost of Administration: This cost is equivalent to the time and effort expended in preparing a purchase order. Consider the vegetable vendor example. The vendor would need to spend a good amount of time determining the amount and type of

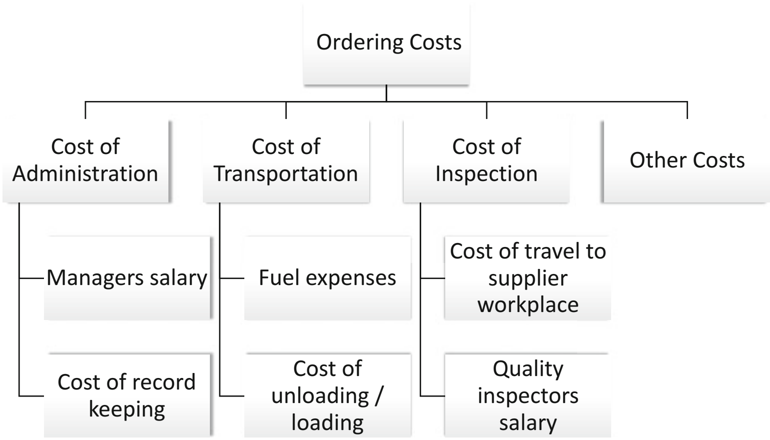

Fig. 2.5 Breakup of ordering costs

vegetables and fruits he would need to purchase the next day. He would need to consider factors like the shelf life of the item as well as the amount already available in stock, besides an estimate of the likely demand. He would also spend some time finding the right supplier as well as negotiating a good price. All this effort and time is part of the administrative cost.

- Cost of Transportation: It would cost the vendor to ship the procured items from the local wholesale market to his retail outlet. This includes cost of loading and unloading.
- Cost of Inspection: There is a possibility that some of the items may get damaged in transit. On receipt of materials at his retail outlet, the vendor and his assistant would inspect and segregate the items. This time and effort would equate to the cost of inspection.
- Other Costs: Certain events may require the vendor to expedite the procurement process. The cost incurred for expediting is also part of ordering cost. Some organizations may have computer systems to maintain inventory of items. This cost of recordkeeping is also part of ordering process.

In this book, the ordering cost is represented by C_o. Figure 2.5 shows a sample breakup of the ordering costs, while Table 2.3 illustrates a typical computation of ordering costs per order for an organization. As can be seen from Table 2.3, the total expenses toward placing orders in a given period amount to \$165,245. Since the number of orders placed (i.e., generated by the purchase department) in that period is 3406, the ordering cost is set as \$165,245/3406 = \$48.5 per order.

Table 2.3 Ordering cost calculation

Expense head	Total annual expenditure ($)	Weightage	Apportioned to ordering cost computation
Cost of administration			
Stationery (e.g. order forms)	1200	10%	120
Communication (telephone, courier, etc.)	4500	25%	1125
Salary – Manager (purchase department)	55,000	80%	44,000
Salary – Recordkeeping assistant (purchase department)	48,000	100%	48,000
Cost of inspection			
Travel to suppliers' workplace (for inspection)	6000	100%	6000
Salary – incoming goods inspector	42,000	100%	42,000
Cost of transportation			
Transportation costs (fuel expenses)	12,000	100%	12,000
Unskilled labor (loading/unloading) – Two persons @ $10 per hour for 2 h per day	12,000	100%	12,000
Total expenses			165,245
Number of orders placed during the year			3406
Order cost per order			$48.5

Adapted from Mahadevan (2015)

2.3.3 Shortage Costs

The cost incurred by an organization when it is unable to satisfy a demand, a situation referred to as a stockout, is called shortage cost.[5] Two scenarios may be possible in the event of a stockout:

- The demand may be back-ordered: Let us go back to the vegetable vendor example. A customer arrives at the retail outlet to buy a specific vegetable. Unfortunately, the vendor has run out of stock of that item. It is possible that the customer may be willing to wait and ask the vendor to supply the same later in the day. In this case, the vendor would have to either (a) place a new order and ask his regular supplier to expedite the same, or (b) procure this item from another supplier at a higher price in the local market and deliver it to the customer. The additional cost incurred in these situations can be considered to be the cost of back-ordering.
- The sale would be lost: In this situation, the customer requesting the specific vegetable would not be willing to wait, and instead go to a competitor. This would result in a loss of profit for the vendor.

[5]The shortage cost is also referred to as penalty cost or the stockout cost.

Table 2.4 Data for computing shortage cost

Date	Ordered quantity	Beginning inventory	Demand	Ending inventory
Day 1	12	12	10	2
Day 2	12	14	13	1
Day 3	12	13	12	1
Day 4	12	13	14	−1
Day 5	12	11	10	1
Day 6	12	13	10	3
Day 7	12	15	13	2

In this book, the shortage cost is represented by C_s. Let us now look at a simple method to compute shortage cost (Naddor 1966). Table 2.4 shows the data necessary for computation of shortage costs.

Let us assume that the vegetable vendor procures exactly 12 kg of tomatoes each day from the wholesaler. Table 2.4 shows the demand, beginning inventory and ending inventory of tomatoes, in kg. Notice the stockout (of 1 kg) on Day 4 in the week. Average shortage during the year is therefore $\frac{1}{7} \times 52 = 7.4$ kg. If the cost of incurring shortages is \$2 per kg per year, then the annual shortage cost.

$$C_s = 7.4 \times 2 = \$14.8 \text{ per year.}$$

The solution assumes that the demand is back-ordered. What is presented here is a very simplified solution. In reality, computation of the cost of incurring shortages is extremely complex.

2.4 Running Example: Managing Inventory at Rosettas Tortilleria

In this section, we present a running example (case study), parts of which would be used in each of the following chapters.

Rosettas Tortilleria ("Rosetta's") is a family-run business in the southern part of Leon, Mexico. They produce a range of food products. Established in 1985, they have seen their market share grow rapidly. Currently, their market share is around 8%. One of the reasons they are popular in Leon is, unlike their competitors, they do not add any preservatives. They have recently set up a large 8000 sq. ft. kitchen and are equipped with very efficient equipment. During the initial years, they did business like anyone would – based on intuition. They have now grown to a stage where a regime of strict control over materials and processes is the only way to manage the business. They have, therefore, hired the right kind of people and are keen they implement scientific inventory control at the earliest.

Tortillas is the staple food in Mexico. Rosetta's produces different varieties of tortillas, but their fast-moving product is the corn-flavored tortilla. More than 30%

Table 2.5 Raw materials used by Rosetta's

S. No.	Item name	Unit price ($)	Unit	Annual demand
1	Vegetable oil	20	Liter	7200
2	Corn flour	30	Kilogram	25,000
3	Butter	65	Kilogram	500
4	Water	0.5	Liter	25,000
5	Cornmeal additive	70	Kilogram	750
6	Salt	6	Kilogram	100
7	Vanilla essence	300	Milliliter	120
8	Eggs	25	Dozens	240
9	Cashew-nut paste	400	Jar	120
10	Printed polybag	8	100 numbers	1200
11	Yeast (imported)	200	Pounds	100
12	Pepper	300	Kilogram	36
13	Goat cheese	500	Kilogram	24
14	Liqueur	200	Liter	120
15	Cooking soda	45	Kilogram	100
16	Cooking gas	4600	Full tank (300 liters)	48

of their revenues come from just this product. Inventory managers at Rosetta's manage a large number of materials ("items") that go into the production of food products. Characteristics of a sample of 16 key items used at Rosetta's are shown in Table 2.5.

Inventory managers at Rosetta's review the stock of items in the stores continuously. Because of their investment in computers and technology, they have up-to-the-minute information on stock levels of each of the items. The characteristics of an item in the list are very different from the others in terms of demand pattern, lead times, costs, etc. The managers know that each of these items would need to be managed differently. An in-depth analysis of some of these items is presented in the following sections:

2.4.1 *Steady, Constant Demand Items*

Rosetta's consumes 7200 liters of vegetable oil each year in production of food products, including tortillas. (This is roughly 20 liters each day, assuming they work 360 days a year.) The demand for vegetable oil is fairly stable and steady each month. They buy vegetable oil from Oxxa, their prime supplier. On an average, it costs Rosetta's $80 to place an order. Accountants at Rosetta's use an inventory carrying rate of 30% per year. Oxxa has the ability to supply the entire ordered quantity in one lot. The storage area in Rosetta's is small. On some occasions, managers have not been able to fit in all the incoming materials in the designated storage area. Oxxa, however, provides an option to gradually deliver up to 24 liters of oil each day.

Given the situation, what inventory strategy should Rosetta's adopt to manage this item? What quantity of vegetable oil should they order in one lot, and when? Should they ask their Oxxa to deliver the complete order in one lot, or should they ask them to deliver the order gradually? Answers to these questions will be addressed in Chap. 3.

2.4.2 Taking Advantage of Supplier Discounts

Month over month, Rosetta's uses large quantities of vegetable oil for production of food products. Vegetable oil is one of the more expensive ingredients in the production process. To improve profitability, procurement personnel at Rosetta's are always looking at minimizing the costs of this item. If Rosetta's procures vegetable oil from the local supermarket on a need basis, it would cost them an average of $20 per liter. Considering the fact that Rosetta's have been a high-demand loyal customer for several years, the local supplier – Oxxa – informs Rosetta's that if they place an order for 500 liters or more, Oxxa would supply vegetable oil at $19.5 per liter.

Also, on certain occasions, Oxxa runs special campaigns to clear off their current stock in anticipation of new, fresh stock of vegetable oil. For a fixed period, they offer a discount of 15% per liter to those buyers that can place an order for larger than usual quantities, before end of the month.

What should Rosetta's procurement strategy be? Should they continue procuring based on their calculated EOQ, or should they take advantage of the discount and order more? These are the questions that will be addressed in Chap. 4.

2.4.3 Items with Time-Varying Demand

Corn flour additive, a customized *masala*,[6] is used by Rosetta's to make corn-flavored tortillas. Rosetta's buys this each month from their preferred vendor. The demand for this item varies each month. It peaks during primavera and starts dipping as summer approaches. It increases once again during the rainy season.

[6] A mixture of spices.

However, the demand for this item is known at least 6 months in advance, thanks to Rosetta's demand estimation system for this product that forecasts fairly accurately the monthly demand. The demand (in kilograms) over the next 6 months (January through June) is as shown below:

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (kg)	36	60	85	11	39	75

Considering the fact that demand varies each month, what quantity should Rosetta's order, and when?

> In Chap. 5, we discuss solutions to manage a single item whose demand varies with time. Popular heuristics such as Lot-for-Lot, Part-Period Balancing, Silver-Meal, Least Unit Cost, and Wagner-Whitin will be discussed.

2.4.4 *Items with Uncertain Demand*

At Rosetta's, demand for vegetable oil has been fairly steady and constant, and that for corn flour additive has been predictable. But the demand for eggs has been very uncertain. There have been instances when less than 200 eggs have been consumed in a week. There have also been instances when more than 900 eggs have been used up in a week. On an average, the demand for eggs has been around 500 per week.

> Because the demand for the item is not constant, the EOQ formula cannot be applied. How, then, would managers at Rosetta's manage inventory of items whose demand is uncertain? This concern will be addressed as part of Chap. 6.

2.4.5 *Management Under Constraints*

Inventory managers at Rosetta's manage a large number of items in stock such as corn flour, vegetable oil, cooking gas, yeast, and several others. Stocking decisions of these items are dependent on each other. For example, the decision to stock a certain amount of vegetable oil depends on the amount of corn flour being stocked. What this means for Rosetta's is that stocking large quantities of vegetable oil and small amount of corn flour is not really going to help. Another problem being faced by them is the fact that the management wants to restrict the amount of money they lock in inventory. In these situations, what should be their replenishment policy?

In Chap. 7, we introduce the concept of multi-item inventory management and discuss simple analytical solutions to inventory problems that are subject to constraints. Specifically, we discuss problems that are subject to budget, space, and number of orders constraints.

Monitoring and controlling of inventory items is very expensive. In a multiproduct inventory system, all items held in stock are not equally profitable. An item with a purchase price of $200 may be considered high-value, but may be used sparingly. The purchase price of another item may be $0.50, which is considered relatively low-value. However, this item may be used in large quantities. Table 2.5 shows a sample of 16 items used by Rosetta's. Which of the items would the inventory manager focus on more?

In a multi-item inventory management system, stocking decision of one item impacts another item. In Chap. 8, we discuss simple analytical and graphical methods of managing multiple items together.

2.4.6 Managing Perishable Items

Corn tortillas are sold to retail customers through their only sales outlet, managed by Maria Fernanda. Tortillas are sold in packs of 10. Each pack of 10 costs $10, and they sell it to customers at $25. Because they do not add any preservatives, the shelf life of a pack of tortilla is 1 day (24 h). Maria accepts a predetermined number of corn-flavored tortilla packs each morning at 7:00 am when the sales outlet opens. At 11:00 pm when the outlet closes, she discards all the unsold packs. She also informs the kitchen manager the number of packs she would need the next morning. One of her key performance indicator (KPI) is to minimize the losses due to unsold inventory, and for this she relies on historical data. Maria must know how many packets she would need at the beginning of the day, each day. Weekly demand for corn tortillas is as shown in Table 2.6.

Table 2.6 Weekly demand for Tortillas

Lower bound	Upper bound	Frequency	Lower bound	Upper bound	Frequency
0	100	2	501	600	30
101	200	7	601	700	35
201	300	12	701	800	28
301	400	18	801	900	15
401	500	23	901	1000	6

Maria does not like wasting food. Nobody does. She has approached a food-processing unit in León that reprocesses food items. The reprocessing unit has agreed to buy unsold packets of tortilla from Rosetta's at $5 per packet. Maria would now have to analyze the new situation and decide on the number of packets she would need each morning when the outlet opens for sale.

> Solutions to problems involving items that have a finite life, with and without salvage value, are discussed in Chap. 9. These are also referred to as single-period inventory models.

2.4.7 *Production Infrastructure*

Rosetta's uses large automated tortilla-making machines. Each machine includes a dough loading section, a pressing section, a spiral heating (carousel) section, and a packaging section. The spiral heating section is driven by a powerful three-phase, 230 V electric motor that is used 365 days a year for 16 h each day. In the event of a motor failure, production losses incurred by Rosetta's would be $100 per hour. The failure rate (p) of the motor is 2 per year, and the mean time to repair (T) is 3 months. Rosetta's has four such identical tortilla-making machines, and currently have one spare rotable motor in their stores.

> How many spare motors must Rosetta's maintain in their inventory? Is one spare motor held in inventory sufficient for Rosetta's to minimize their production losses? What would be the savings in downtime if they employ one more spare motor? These are some of the questions that would be addressed in Chap. 10.

2.4.8 *Distribution to Retailers*

To increase their market share, Rosetta's has now decided to allow a sole franchisee in León to sell the Rosetta's brand of tortillas. Franchisee places an order for a certain number of packs of tortillas. This costs them $4 per order. Rosetta's supplies each pack of tortilla to the franchisee at a price of $10. The franchisee can retail a pack of tortilla at $22. Inventory carrying rate is 30% per annum, and the annual demand for tortilla is 6000 packs. Rosetta's "incurs" an order cost of $8 per order when they procure the packets from their kitchen. Also, we assume the tortillas have an indefinite shelf life.

In Chap. 11, we address the inventory management problem in a simple two-stage multi-echelon situation. We develop equations to determine order quantities for each stage.

2.5 Summary

In this chapter, we discussed key design factors of inventory control system including frequency of review, timing of order placement, and order size. These factors need to be kept in mind while adopting an inventory control system for any given business. We reviewed three popular inventory control systems in this chapter:

- Continuous Review, Fixed Order Quantity (s, Q) System
- Continuous Review, Order-Up-to-Level (s, S) System
- Periodic Review, Order-Up-to-Level (T, S) System

We also discuss in detail three relevant inventory costs – carrying cost, ordering cost, and shortage cost – as well as methods to estimate those.

Toward the end of this chapter, we presented a detailed case study – Rosetta's Tortilleria – that would be used as a running example across all chapters. This would help get a common basis for learning inventory management concepts.

2.6 Practice Problems

Problem 2.1
Table 2.7 shows data collated for a single item in an inventory system. Compute the following:

(a) Inventory holding rate
(b) The annual carrying cost of an item that costs $25
(c) The total annual carrying cost for 15 items that cost $25 per unit
(d) The total carrying cost for 15 items that cost $25 per unit and are held in inventory for a period of 5 years.

Table 2.7 Inventory Data for Problem 2.1

Type of cost	Values (per year)
Cost of capital	9.25%
Cost due to breakage	6.25%
Cost of physical handling	0.50%
Rent paid toward physical space for storage	3.50%
Premium paid to insure inventory	0.50%
Tax	1.00%

Table 2.8 Data for Problem 2.2

Expense head	Total annual expenditure ($)	Weightage	Apportioned to ordering cost computation
Cost of administration			
Stationery (e.g. order forms)	1000	5%	
Communication (telephone, courier, etc.)	5500	25%	
Salary – Manager (purchase department)	65,000	85%	
Salary – Recordkeeping assistant (purchase department)	48,000	100%	
Cost of inspection			
Travel to suppliers' workplace (for inspection)	3500	100%	
Salary – Incoming goods inspector	50,000	100%	
Cost of transportation			
Transportation costs (fuel expenses)	15,000	100%	
Unskilled labor (loading/unloading) – Two persons @ $10 per hour for 2 h per day	18,000	100%	
Total expenses			
Number of orders placed during the year			
Order cost per order			

Answer

(a) Inventory holding rate is 21%
(b) The annual carrying cost for one item is $5.25
(c) Total annual carrying cost for 15 items is $78.75
(d) Total annual carrying cost for 15 items held for 5 years is $393.75

Problem 2.2
Table 2.8 shows costs incurred by an organization during one financial year. Fill up the last column in the table. Also, if the organization has placed 5000 orders in the financial year, compute the average cost per order.

Answer

- Total expenses that can be charged to ordering is $191,175
- If 5000 orders were placed, the cost per order is $38.23

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Part II
Single-Item Inventory Models

Chapter 3
Deterministic Inventory Models

3.1 Introduction to Economic Order Quantity (EOQ) Model

Rosetta's consumes 7200 liters of vegetable oil each year in the production of food products, including tortillas. (This is roughly 20 liters each day, assuming they work 360 days a year.) The demand for vegetable oil is fairly stable and steady each month. They buy vegetable oil from Oxxa, their prime supplier. On an average, it costs Rosetta's $80 to place an order. Accountants at Rosetta's use an inventory carrying rate of 30% per year. Oxxa has the ability to supply the entire ordered quantity in one lot. The storage area in Rosetta's is small. On some occasions, managers have not been able to fit in all the incoming materials in the designated storage area. Oxxa, however, provides an option to gradually deliver up to 24 liters of oil each day.
Given the situation, what inventory strategy should Rosetta's adopt to manage this item? What quantity of vegetable oil should they order in one lot, and when? Should they ask their Oxxa to deliver the complete order in one lot, or should they ask them to deliver the order gradually? Answers to these questions will be addressed in this chapter.

The Economic Order Quantity (EOQ) model is the most basic of all inventory models that helps inventory manager answer the question "*how much should I order*"?. This model is based on the philosophy of determining an order quantity where the sum of the ordering costs and the carrying costs is minimal.

D. Shenoy, R. Rosas, *Problems & Solutions in Inventory Management*,
DOI 10.1007/978-3-319-65696-0_3

3.1.1 Assumptions

Following are some of the assumptions made in deriving an equation for an EOQ:

- We consider a single item in the inventory system. The demand for this item is known and is fairly constant.
- The policy is to review the inventory system continuously. A fixed-size replenishment order is placed as soon as the inventory of the item reaches the reorder level, that is, (s, Q) model is in operation.
- The replenishment occurs instantaneously. All items ordered are received at the same time, in full, and in good quality.
- The ordering cost, holding cost, and the unit price remain constant. No discounts are involved.
- Because the replenishment occurs instantaneously, there are no shortages. Also, there is no need for back-ordering.

Please note that some of the above assumptions will be relaxed later in this chapter or in the subsequent chapters.

3.1.2 EOQ Derivation

Let us now derive an expression that will help us determine an economic order quantity that answers the question of *how much to order* given that the assumptions listed above are satisfied. We consider a planning horizon of 1 year. The total annual inventory cost (TIC) is the sum of ordering costs, holding costs, and the purchasing costs:

$$\begin{aligned}\text{TIC} = &\ \text{Annual ordering costs} + \text{Annual holding costs} \\ &+ \text{Annual purchasing costs}\end{aligned} \tag{3.1}$$

The relationship between these costs is shown in Fig. 3.1. Let us formulate mathematical expressions for each of the terms (Anderson et al. 2016; Nahmias 2005). Let Q be the economic order size. If the annual demand for the item is D units, then the number of orders to be placed, N, in a year is

$$= \frac{D}{Q} \tag{3.2}$$

If the ordering cost is C_o per order, the annual ordering cost is

$$= \frac{D}{Q} C_o \tag{3.3}$$

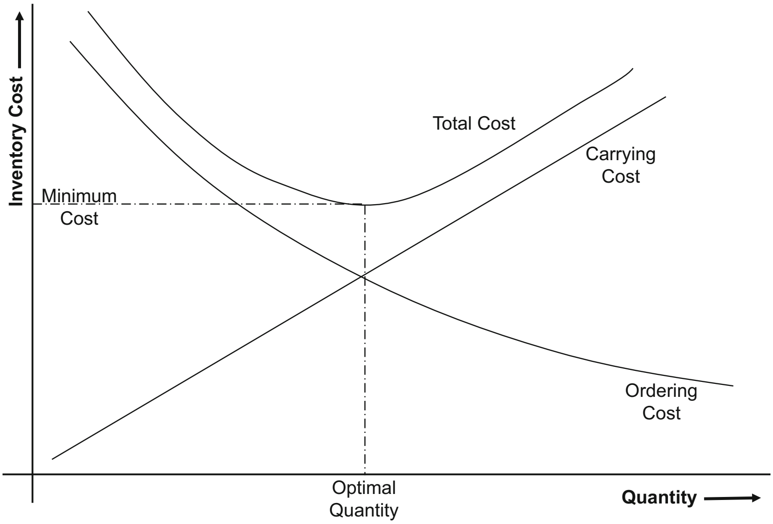

Fig. 3.1 Total inventory cost function

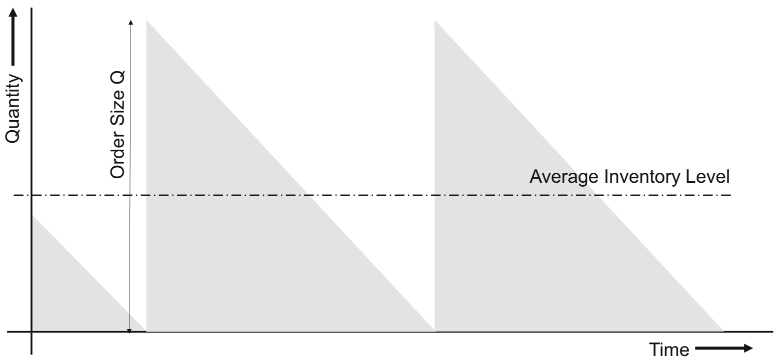

Fig. 3.2 Inventory level as a function of time

If we place an order for Q units and the order is filled instantaneously, then the beginning inventory in the cycle is Q units. We place the next order only when the inventory is completely exhausted. This is shown in Fig. 3.2. Since the maximum inventory is Q units and the minimum inventory is 0, the average inventory in that cycle is

$$= \frac{Q + 0}{2} \tag{3.4}$$

There may be several identical cycles in a planning horizon of 1 year, and because the average inventory in a cycle is $\frac{Q}{2}$, the average inventory over the entire duration of planning horizon is also $\frac{Q}{2}$. If the cost of the item is C per unit and the inventory holding rate is i per year, the annual holding cost is

$$= \frac{Q}{2} iC \tag{3.5}$$

Also, the annual purchasing cost is

$$= DC$$

Substituting the above terms in Eq. 3.1, we obtain an equation for the TIC, which is

$$\text{TIC} = \frac{D}{Q} C_o + \left(\frac{Q}{2}\right) iC + DC \tag{3.7}$$

To determine the optimal order quantity, we differentiate Eq. 3.7 with respect to Q and equate it to 0. By doing so, we get

$$\frac{d(\text{TIC})}{dQ} = -\frac{D}{Q^2} C_o + \frac{iC}{2} = 0$$

Simplifying the above, we get

$$Q = \sqrt{\frac{2DC_o}{iC}} \tag{3.8}$$

where Q is the optimal order quantity, also referred to in this book as EOQ. Note the following:

- The term iC can also be represented by C_h. In other words, $C_h = iC$.
- The term for purchase cost is not part of the final equation, since that term is independent of Q.

Once we obtain Q, we may need to also look at the timing of placement of successive orders. This is called the cycle time, represented by τ, and can be determined by

$$\tau = \frac{Q}{D} \tag{3.9}$$

3.1.3 EOQ Dimensions

Let us now examine the physical dimensions involved in the EOQ equation (Eq. 3.8):

- The demand, D, is expressed in terms of number of units per year.
- The ordering cost, C_o, is expressed in terms of dollars
- The interest rate, i, is given in % per year.
- The unit price (cost), C, of the item is expressed in dollars per unit.

Using the above, we get

$$Q = \sqrt{\frac{\left(\frac{\text{units}}{\text{year}}\right) \times \text{dollars}}{\left(\frac{1}{\text{year}}\right) \times \left(\frac{\text{dollars}}{\text{unit}}\right)}} = \sqrt{\left(\frac{\text{units}}{\text{year}}\right) \times \text{dollars} \times \left(\frac{\text{year}}{1}\right) \times \left(\frac{\text{unit}}{\text{dollars}}\right)}$$

Simplifying the above, we see that the physical dimension for Q is (number of) units (Monks 1987). It is important that the *time units* for D and C_h must be the same. For example, if the demand is given in units per year, then the holding cost must also be in dollars per unit per year.

3.1.4 TIC Computation: Alternative Method

We can use another method to determine the TIC once we have the EOQ. If we ignore the annual purchasing cost in Eq. 3.7, we have

$$\text{TIC} = \frac{D}{Q} C_o + \frac{Q}{2} iC \tag{3.10}$$

Substituting the value of Q from Eq. 3.8 in Eq. 3.10, we get

$$\text{TIC} = \frac{D}{\sqrt{\frac{2DC_o}{iC}}} C_o + \frac{\sqrt{\frac{2DC_o}{iC}}}{2} iC$$

On simplification, we get

$$\text{TIC} = \sqrt{2C_o DiC} \tag{3.10a}$$

Either of Eq. 3.7, Eq. 3.10, or Eq. 3.10a may be used to determine TIC. It should be noted that Eqs. 3.10 and 3.10a do not include the cost of purchase.

Let us now apply the EOQ concept to the running example presented at the beginning of this chapter. Rosetta's uses a large quantity of vegetable oil for production of food products. The demand for this item is fairly steady. The objective is to determine the economic order quantity. We have the following information with us:

- Order cost C_o is \$80 per order.
- The unit price (C) of vegetable oil is \$20 per liter. The inventory holding rate (i) is 30% per year.
- The demand is fairly constant at 7200 liters per year.

The first step is to verify that the time units for demand and holding cost are the same. In this case, it is the same (both are expressed in years). Therefore, we can progress to the next step. Substituting the above values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 20}} = 438.17 \text{ liters}$$

Thus, the EOQ is 438.17 liters. It should be noted that the demand for the item is continuous. This implies that the EOQ may be a non-integer, and this is perfectly acceptable. The TIC can be calculated using Eq. 3.10. Substituting the values in Eq. 3.7, we get

$$\text{TIC} = \left(\frac{7200}{438} \times 80\right) + \left(\left(\frac{438}{2}\right) \times (0.3 \times 20)\right) = \$2629$$

The TIC is *\$*2629. It is to be noted that this cost does not include the cost of investment (cost of purchase).

A few numerical problems are presented that will help reinforce the concepts learnt in this section.

Solved Problem 3.1

Leon Cardiology Centre in Mexico buys 25,000 stents each year from its suppliers in Germany. Each stent costs \$1500, and carrying cost is 26% of the value of the average inventory of stents per year. If the ordering cost is \$270 per order, determine the economic order quantity for stents. Also, determine the number of orders and the TICs.

Solution

In this problem, the demand and carrying cost units are the same. Other information we have are as follows:

- D is 25,000 per year.
- C is \$1500 per stent.
- i is 26% per year, or 0.26.
- C_o is \$270 per order.

Substituting the values in the EOQ formula (Eq. 3.8), we get

$$Q = \sqrt{\frac{2 \times 25000 \times 270}{0.26 \times 1500}} = 186 \text{ stents}$$

The cardiology center needs to place an order for 186 stents each time they place an order.

The number of orders can be determined using Eq. 3.2. Substituting the values in Eq. 3.2, we get

$$N = \frac{D}{Q} = \frac{25000}{186} = 134 \text{ orders per year}$$

The number of orders that need to be placed is 134 per year.

The TICs can be determined using Eq. 3.10. Substituting the values in Eq. 3.10, we get

$$\text{TIC} = (134 \times 270) + \left(\frac{186}{2} \times 0.26 \times 1500\right) = \$72,560$$

The TICs (ignoring the cost of investment) is $72,560.

Solved Problem 3.2

SleepWell Mattresses manufactures high-quality spring-based cotton mattresses. A set of eight identical stainless steel springs are used to produce a mattress. The inventory holding cost for springs is $2.15 per spring per year. SleepWell has estimated an annual demand for 20,000 mattresses. Determine the quantity of springs SleepWell should procure to minimize the TICs for springs if the ordering cost per order is $50? Also, compute the average inventory level.

Solution

In this problem,

- D is 20000 × 8 = 160,000 per year. (8 springs are required per mattress).
- iC is $2.15 (the carrying cost is directly specified in this problem).
- C_o is $50 per order.

Substituting the values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 160000 \times 50}{2.15}} = 2728 \text{ springs}$$

SleepWell needs to procure 2728 springs per order.

The average inventory level can be found using Eq. 3.4. Substituting the values, the average inventory level in a cycle is

$$= \frac{Q}{2} = \frac{2728}{2} = 1364 \text{ springs}$$

Solved Problem 3.3

Compute the economic lot size for an item that has an annual demand of 5000 units. Assume the inventory holding costs are based on an annual interest rate of 20%. Further, the purchase cost of the item is $10 and the ordering cost is $25.20 per order. Also, compute the cycle time if there are 250 workdays in a year.

Solution

In this problem, the time units for demand and carrying cost are the same. Other information provided are as follows:

- D is 5000 per year.
- i is 0.20.
- C is $10 per item.
- C_o is $25.20 per order.

Substituting the values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 5000 \times 25.20}{0.2 \times 10}} = 355 \text{ units}$$

The EOQ for the item is 355 units.

The cycle time, τ, can be determined using Eq. 3.9. Substituting the values for Q and D in Eq. 3.9, we get.

$$\tau = \frac{Q}{D} = \frac{355}{5000} = 0.071 \text{ years.}$$

Since there are 250 days in a year, the cycle time is $0.071 \text{ years} \times 250 \ \frac{\text{days}}{\text{year}} = 17.8 \text{ days.}$

Solved Problem 3.4

Sun Corporation is a retailer of school notebooks. They buy notebooks from a wholesaler at $0.50 and sell it to consumers at $0.85 per notebook. The demand for notebooks is estimated at 9000 per quarter. If the ordering cost is $4 per order and carrying cost is based on an annual interest rate of 15%, compute the economic order size.

Solution

In this problem, the time unit for D is quarter, while the time unit for carrying cost (rate) is year. We therefore need to convert the demand from quarter to year. This can be done by multiplying the quarterly demand by 4. Thus, the demand $D = 9000 \times 4 = 36{,}000$.

Further,

- i is 0.15.
- C is $0.50 per notebook. Note that we do not need the selling price.
- C_o is $4 per order.

Substituting the above values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 36000 \times 4}{0.15 \times 0.50}} = 1960$$

Sun Corporation needs to procure 1960 notebooks per order.

Solved Problem 3.5
Based on the following data for an item, what lot size would be economical to procure?

- Usage: 2000 units per year
- Purchase cost of the item: $2
- Inventory holding rate: 25% per annum
- Fixed delivery charges and cost of receiving goods: $5 per shipment

Solution
The following information are available:

- D is 2000 per year.
- i is 0.25.
- C is $2 per item.
- C_o is $5 per order. Order cost includes the cost of delivery charges and cost of receiving goods/shipments.

Substituting the above values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 2000 \times 5}{0.25 \times 2}} = 200 \text{ units}$$

The EOQ is 200 units.

Solved Problem 3.6
The EOQ for an item is 150 units and its annual demand is 2400 units. If the ordering cost per order is $20 per order, compute the implied carrying cost for this item.

Solution
The EOQ is given by

$$Q = \sqrt{\frac{2DC_o}{iC}}$$

In this problem, we have been given the EOQ which is 150 units, and we need to determine the carrying cost, iC

$$iC = \frac{2DC_o}{Q^2} = \frac{2 \times 2400 \times 20}{(150)^2} = \$4.26 \text{ per unit per year}$$

The carrying cost is $4.26 per unit per year.

3.2 When to Order: Incorporating Lead Time

In the previous section, we discussed a solution to the question of "*how much to order?*". This section describes the solution for "*when to place an order?*". In the basic EOQ model, we assumed the order is filled instantaneously. In other words, the model assumed the procurement lead time[1] is zero. This is not true always. In this section, we relax the lead time assumption to make the model more realistic.

At the time of placing an order, the level of inventory on hand should be such that it can satisfy the demand during the lead time. Clearly, this level of inventory is a function of the consumption during lead time. This level, also referred to as reorder level,[2] s, can be mathematically expressed as

$$s = L \times d \tag{3.11}$$

where L is the lead time, in days, and d is the daily demand. In Rosetta's running example case, if the procurement lead time for vegetable oil is 7 days, the reorder level is

$$s = 7 \times \frac{7200}{360} = 140 \text{ liters}$$

Here, we assume Rosetta's is working 360 days a year. The inventory control policy for Rosetta's based on a continuous review, fixed order quality (s, Q) system can be stated as follows (see also Fig. 3.3):

> When the on-hand inventory level of vegetable oil goes down to 140 liters we place an order for 438 liters.

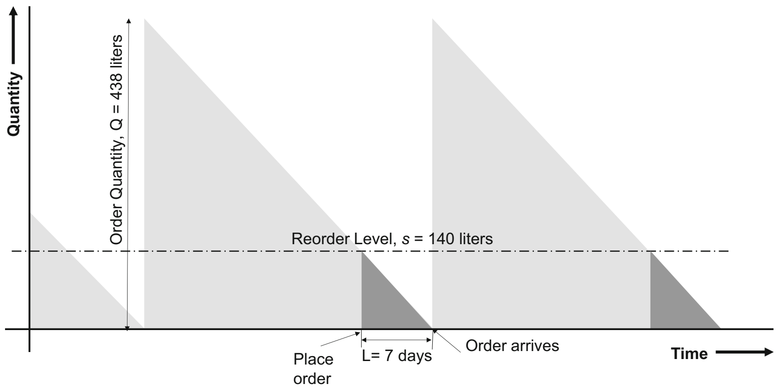

Fig. 3.3 Replenishment policy for Rosetta's based on (s, Q) system

[1]Procurement lead time is time between placing an order and receipt of the same.

[2]Also called reorder point.

Solved Problem 3.7
A firm sells an item that has an annual demand of 1000 units. If the procurement lead time is a constant 5 days, find the reorder level. Assume 365 workdays a year.

Solution
Since the annual demand is 1000 units, and the firm works 365 days a year, we can compute the daily demand, which is

$$= \frac{1000}{365} = 2.74 \text{ units per day}$$

Substituting the values in Eq. 3.11, we get

$$s = 5 \times 2.74 = 13.7 \text{ units}$$

The reorder level is 13.7 (14 units if we round off to the next highest integer) units. When the on-hand inventory level of the item goes down to 14 units, we place a new replenishment order.

A Note on (s, S) Inventory Model
Recall from our discussions in Chap. 2 that (s, S) is one where the inventory level of an item is reviewed continuously. An order is placed when the inventory level drops to s. The size of the order is such that it would take the inventory level back up to level S. A key difference between (s, Q) and (s, S) is that the order size in the latter model is variable. However, consider the following situation: if we assume that all demands for the item are of unit size, then an order is placed exactly when the inventory level falls to s. In this case, the size of all the orders would be the same, given by $Q = S - s$. It should be noted that this formula may be used *only* when all transactions (demand) are of unit size (Silver et al. 1998).

3.3 EOQ Model with Gradual Replenishments

Another assumption made in the derivation of the classic EOQ was that the ordered quantity arrives in full. For example, Rosetta's places an order for 438 liters of vegetable oil, and all of it arrives in one instance. This may not be true in some cases as the supplier may deliver parts of the order at different times. It may also be that Rosetta's (the purchaser) does not want to receive the whole order in one instance due to space constraints they may have in their warehouse. In this section, we will learn about the case where the supplier delivers the ordered material at some uniform rate rather than all in one lot. Figure 3.4 illustrates the concept of gradual supply (Gaither 1987).

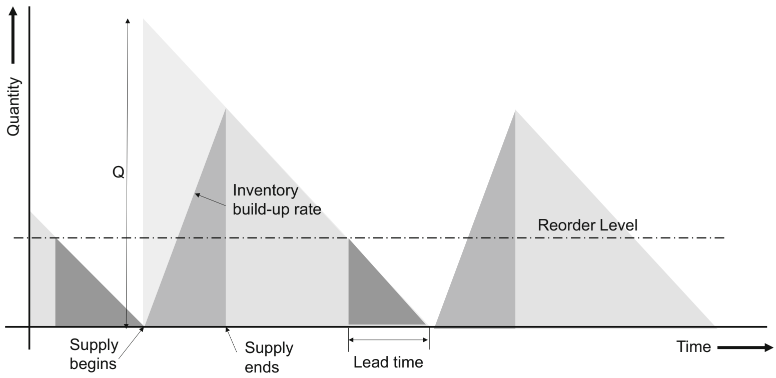

Fig. 3.4 Inventory model – gradual supply

If d is the rate at which items are consumed, and p is the rate at which items are supplied, the inventory build-up rate (IBR) is given by

$$\text{IBR} = \left(\frac{p-d}{p}\right)$$

One of the assumptions we make here is that p is always greater than d. The maximum inventory level is therefore

$$\left(\frac{p-d}{p}\right)Q$$

Since the minimum inventory level is zero, the average inventory level can be expressed as follows:

$$= \frac{\left(\frac{p-d}{p}\right)Q + 0}{2}$$

Since

$$\begin{aligned}\text{TIC} = {} & \text{Annual ordering costs} + \text{Annual holding costs} \\ & + \text{Annual purchasing costs}\end{aligned}$$

we have

$$\text{TIC} = \frac{D}{Q}C_o + \left(\frac{p-d}{p}\right)\frac{Q}{2}iC + DC \tag{3.12}$$

To obtain the minima, we differentiate the above with respect to Q and equate it to 0. By doing so, we get

$$\frac{d(\text{TIC})}{dQ} = -\frac{D}{Q^2}C_o + \left(\frac{p-d}{p}\right)\frac{iC}{2} = 0$$

or

$$Q = \sqrt{\frac{2DC_o}{iC}\left(\frac{p}{p-d}\right)} \tag{3.13}$$

We can now use Eq. 3.13 to solve Rosetta's order quantity problem (see Box at the beginning of this chapter) if the supplies are delivered gradually at some uniform rate and not all in one lot. Referring to the running example, we know that Oxxa, the supplier, delivers vegetable oil at the rate of 24 liters a day. We also know that the annual demand for vegetable oil is 7200 liters, or daily consumption of vegetable oil is 20 liters. We now have the following information with us that we could use to solve this problem:

- Order cost C_o is $80 per order.
- The purchase price (cost) of vegetable oil (C)is $20 per liter. The inventory holding rate (i) is 30% per year.
- The demand is fairly constant at 7200 liters per year (or demand is 20 liters per day).
- The supplier supplies vegetable oil at the rate of 24 liters per day.

Substituting the above in Eq. 3.13, we get

$$Q = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 20}\left(\frac{24}{24-20}\right)} = 1073$$

The EOQ in this case would be 1073 liters compared to 438 liters if all units are delivered in one go. The TIC can be determined using Eq. 3.12. (It should be noted that we ignore the purchasing cost term.) Substituting the values in the equation, we get

$$\text{TIC} = \frac{7200}{1073} \times 80 + \left(\frac{24-20}{24}\right) \times \frac{1073}{2} \times 0.3 \times 20 = \$1073$$

Table 3.1 summarizes the solution for Rosetta's vegetable oil ordering problem.

By adopting a gradual supplies strategy, Rosetta's will be able to save $1556 every cycle. In percentage terms, the saving would be

$$\frac{2629-1073}{2629} = 59\%\text{savings}$$

Solved Problem 3.8

Consumption of a bought-out item in a manufacturing organization is 100 per day. The supplier supplies this item to the manufacturer at the rate of 300 per day. If the

Table 3.1 EOQ and TIC for Rosetta's inventory management problem

	Order filled immediately	Order filled gradually
EOQ	438 units	1073 units
TIC	$2629	$1073

carrying cost is $0.1 per item per day and the ordering cost is $250 per order, compute the EOQ for this item.

Solution
We have the following data with us:

- Demand: 100 per day
- Supply rate: 300 per day
- Order cost: $250 per order
- Carrying cost: $0.1 per item per day

Note that all the *data supplied is in days*. We can use the data as is. Substituting these values in Eq.3.13, we get

$$Q = \sqrt{\frac{2 \times 100 \times 250}{0.1} \times \left(\frac{300}{300 - 100}\right)} = 866 \text{ units}$$

The economic order quantity is 866 units.

Solved Problem 3.9
You have developed the following estimates for procuring an item for your manufacturing operations:

- Item demand: 3600 units yearly (10 units per day)
- Purchase price: $25 per item
- Ordering cost: $35 per order
- Inventory holding rate: 25% annually

The following two options are available to you:

- Option 1: The supplier can supply all items at once.
- Option 2: The supplier can supply 15 items per day.

Compare the following for each of the options – total ordering cost, total holding cost, cycle time, number of orders, and the TICs.

Based on the information you have, which of the above options would you prefer?

Solution
We solve this problem by computing the EOQ as well as the TIC for each of the options. The option that has minimal TICs is the one that we need to use. Calculations are as shown in Table 3.2.

Since the TIC for Option 2 is cheaper, we would prefer items to be gradually supplied at the rate of 15 items per day.

Table 3.2 Calculations for Solved Problem 3.9

Parameter	Option 1: Supply all items in one lot	Option 2: Supply items gradually at a rate of 15 items per day
EOQ	$Q = \sqrt{\frac{2DC_o}{iC}} = \sqrt{\frac{2 \times 3600 \times 35}{0.25 \times 25}}$ =201 units	$Q = \sqrt{\frac{2DC_o}{iC}\left(\frac{p}{p-d}\right)} = \sqrt{\frac{2\times3600\times35}{0.25\times25} \times \left(\frac{15}{15-10}\right)}$ =348 units
Ordering costs	$= \frac{D}{Q}C_o = \frac{3600 \times 35}{201} = \$627$	$= \frac{D}{Q}C_o = \frac{3600 \times 35}{348} = \$362$
Holding costs	$= \frac{Q}{2}iC = \frac{201}{2} \times 0.25 \times 25 = \$628$	$= \left(\frac{p-d}{p}\right)\frac{Q}{2}iC$ $= \left(\frac{15-10}{15}\right) \times \frac{348}{2} \times 0.25 \times 25$ $=\$363$
Cycle time	$= \frac{Q}{D} = \frac{201}{3600} = 0.055$ years = 20.1 days	$= \frac{Q}{D} = \frac{348}{3600} = 0.097$ years = 34.8 days
Number of orders	$= \frac{D}{Q} = \frac{3600}{201} = 17.9$ orders per year	$= \frac{D}{Q} = \frac{3600}{348} = 10.3$ orders per year
TIC	\$1,255	\$725

3.4 EOQ Model with Planned Shortages

In some cases, businesses allow shortages when the cost of such shortage is manageable (Muckstadt and Sapra 2010). In cases where shortages are allowed, another cost factor becomes important – shortage cost, represented in this section by C_s. When demand arises and is not met, it remains back-ordered (or backlogged) till new stock arrives. A portion of the new stock, when received, is used up to meet the back-ordered demand while the rest builds up the inventory. This means that during a period T_1, inventory exists on hand, and during another period T_2 there is a back-order. If B is the planned back-order and S is the maximum inventory in a cycle, then

$$S = Q - B \tag{3.14}$$

The average inventory held is, therefore,

$$= \frac{S}{2}$$

and the annual inventory carrying cost is

$$= \frac{S}{2}C_h \tag{3.15}$$

Since B is the planned back-order, the average shortage in the system is

$$= \frac{B}{2}$$

If C_s is the shortage cost per unit per year, the annual cost of shortage is given by

$$= \frac{B}{2} C_s \tag{3.16}$$

The TIC is, therefore,

$$\text{TIC} = \frac{D}{Q} C_o + \frac{S}{2} C_h \left(\frac{T_1}{T_1 + T_2} \right) + \frac{B}{2} C_s \left(\frac{T_2}{T_1 + T_2} \right) \tag{3.17}$$

Using the concept of similarity of triangles, we get

$$\left(\frac{T_1}{T_1 + T_2} \right) = \frac{(Q - B)}{Q}$$

and

$$\left(\frac{T_2}{T_1 + T_2} \right) = \frac{B}{Q}$$

Substituting these in Eq. 3.17, we get

$$\text{TIC} = \frac{D}{Q} C_o + \frac{(Q - B)^2}{2Q} C_h + \frac{B^2}{2Q} C_s \tag{3.18}$$

Note from Eq. 3.12, $S = Q - B$.
Partially differentiating the above with respect to B and equating to 0, we get

$$B = \frac{Q C_h}{(C_h + C_s)} \tag{3.19}$$

Partially differentiating the above with respect to Q and equating to 0, we get

$$Q = \sqrt{\frac{2 D C_o (C_h + C_s)}{C_h C_s}} \tag{3.20}$$

Equations 3.19 and 3.20 can be used to determine the back-order size and the optimal order quantity, respectively (Figs. 3.5 and 3.6).

Solved Problem 3.10
WT Accessories sells just one type of all-weather rubber floor mats that are used in cars. Demand for floor mats is 500 each year. The inventory holding cost, C_h,is \$0.75 per piece per year, and ordering cost per order, C_o, is \$20 per order. If back-ordering is allowed and the shortage cost is \$5 per piece per year, compute the EOQ, the number of orders, and the TIC. Also, compute the time over which inventory is on hand and time over which shortages occur.

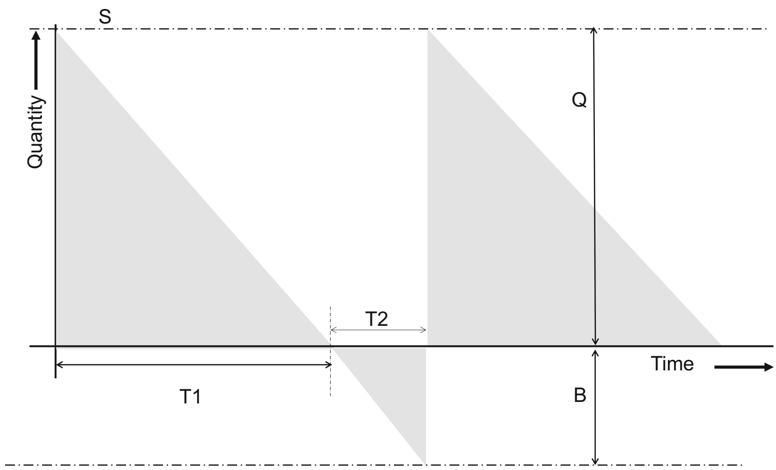

Fig. 3.5 Inventory model with planned shortages

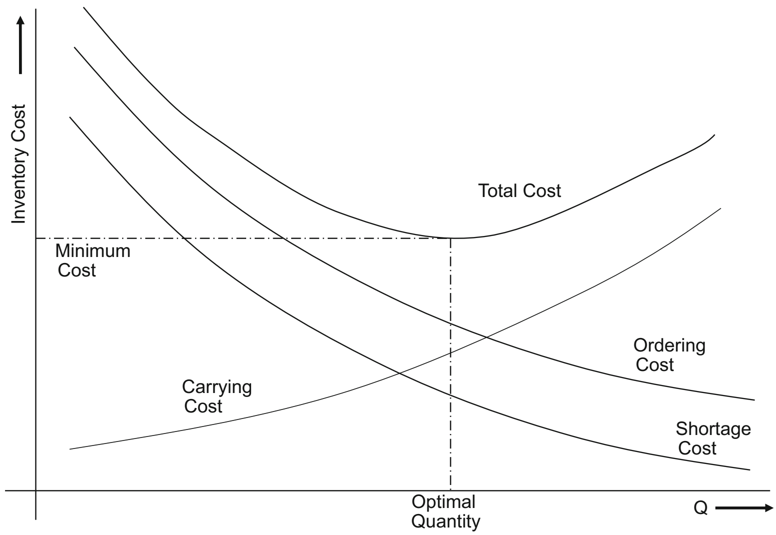

Fig. 3.6 TIC – planned shortages

Solution
Substituting values in Eq. 3.20, we get

$$Q = \sqrt{\frac{2 \times 500 \times 20 \times (0.75 + 5)}{(0.75 \times 5)}} = 175$$

The optimal order quantity is 175 floor mats.

The size of the back-order can be computed using Eq. 3.19. Substituting the values, we get

$$B = \frac{175 \times 0.75}{(0.75 + 5)} = 23$$

The size of back-order is 23 floor mats.

The maximum inventory level S is given by

$$S = Q - B = 152$$

The number of orders per year is

$$= \frac{D}{Q} = \frac{500}{175} = 2.85$$

The TIC can be computed using Eq. 3.18. Substituting the values, we get

$$\text{TIC} = \left(\frac{500}{175} \times 20\right) + \left(\frac{(175 - 23)^2}{2 \times 175} \times 0.75\right) + \left(\frac{23^2}{2 \times 175} \times 5\right) = \$114.21$$

Time during which inventory is on hand is

$$T_1 = \frac{Q - B}{D} = \frac{175 - 23}{500} = 0.30 \text{ years}$$

Time during which shortage occurs is

$$T_2 = \frac{B}{D} = \frac{23}{500} = 0.046 \text{ years}$$

Solved Problem 3.11

MedPlus Pharmacy sells 20 strips of paracetamol tablets each day (annual demand is 7300 strips). MedPlus buys its supplies from its preferred wholesaler at \$1.85 per strip and sells it over the counter to customers at \$2.50 per strip. MedPlus incurs a cost of \$5 every time it places an order. An annual interest rate of 25% is used to determine carrying costs. MedPlus would like to place a standing order with the wholesaler to regularly supply strips of paracetamol tablets. What should be the size of a standing order it should place if:

- The wholesaler supplies all the tablet strips instantaneously in one lot with no back-ordering allowed?
- The wholesaler supplies tablet strips gradually at the rate of 25 strips a day?
- Back-ordering is allowed? (Use a shortage cost of \$2 per strip per year.)

Solution

Case: Wholesaler supplies tablet strips instantaneously
We use Eq. 3.8 to compute the economic order quantity assuming there is no back-ordering, supply is instantaneous, and all tablet strips are delivered in one lot. Substituting the values in Eq. 3.8, we get

$$Q = \sqrt{\frac{2 \times 7300 \times 5}{0.25 \times 1.85}} = 397 \text{ strips}$$

The TICs (ignoring the investment cost) for instantaneous supply in one lot without back-ordering allowed can be computed using Eq. 3.7:

$$\text{TIC} = \left(\frac{7300}{397} \times 5\right) + \left(\left(\frac{397}{2}\right) \times 0.25 \times 1.85\right) = \$183.75$$

Case: Wholesaler supplies tablet strips gradually
We use Eq. 3.11 to compute the economic order quantity assuming there is no back-ordering and supply is gradually at the rate of 25 strips a day. Substituting the values in Eq. 3.11, we get

$$Q = \sqrt{\frac{2 \times 7300 \times 5}{0.25 \times 1.85}\left(\frac{25}{25 - 20}\right)} = 888 \text{ strips}$$

The TICs for the case of gradual supply can be determined using Eq. 3.10. Substituting the values in this equation, we get

$$\text{TIC} = \frac{7300}{888} \times 5 + \left(\left(\frac{25 - 20}{25}\right) \times \frac{888}{2} \times (0.25 \times 1.85)\right) = \$82.18$$

Case: Back-ordering allowed
We use Eq. 3.18 to compute the economic order quantity assuming there is back-ordering and the cost of shortage is \$2 per strip per year.

$$Q = \sqrt{\frac{2 \times 7300 \times 5 \times ((0.25 \times 1.85) + 2)}{(0.25 \times 1.85) \times 2}} = 441 \text{ strips}$$

The TIC for the case where back-ordering is allowed can be computed using Eq. 3.18. Substituting the values in the equation, we get

$$\text{TIC} = \left(\frac{7300}{441} \times 5\right) + \left(\frac{(441 - 83)^2}{2 \times 441} \times (0.25 \times 1.85)\right) + \left(\frac{83^2}{2 \times 441} \times 2\right)$$
$$= \$165.59$$

Table 3.3 Solution summary for Solved Problem 3.11

Decision variable	Supply in one lot, no back-ordering allowed	Gradual supply	Back-ordering allowed
Size of standing order	397 strips	888 strips	440 strips
TIC	\$183.75	\$82.18	\$165.59

As can be seen from Table 3.3, the TIC for the gradual supply is the least expensive. Therefore, the decision would be to place an order for 888 strips.

3.5 Periodic Review Model: Deterministic Demand

All models discussed previously in this chapter are continuous review-based models. In this section, we will discuss a periodic review-based (T, S) inventory model. The decision variables in a periodic review model is the optimal time between inventory review periods (T), and the maximum inventory level (S). (Hillier and Lieberman 2001; Jacobs and Chase 2011). Note that all assumptions we made for the basic EOQ model holds good for this model as well, including the assumption that the demand during lead time is known with certainty.

If T is the optimal time between two orders (inside a year), then the total annual average carrying costs would be

$$= \left(\frac{DT}{2}\right) iC \tag{3.21}$$

and the total annual average ordering costs would be

$$= \left(\frac{D}{DT}\right) C_o \tag{3.22}$$

The TIC would, therefore, be

$$\text{TIC} = \left(\frac{DT}{2}\right) iC + \left(\frac{D}{DT}\right) C_o \tag{3.23}$$

To obtain the minima, we differentiate Eq. 3.23 with respect to T and equate it to 0. Doing so, we get

$$\frac{d(\text{TIC})}{dT} = \left(\frac{D}{2}\right) iC - \left(\frac{1}{T^2}\right) C_o = 0$$

Simplifying, we get

$$T = \sqrt{\frac{2C_o}{iDC}} \tag{3.24}$$

The order quantity is variable and depends on the stock on hand at the time of review and the maximum inventory level, S.

Solved Problem 3.12
An organization has a single item whose annual demand is 15,000 units, unit cost is \$2.5, ordering cost is \$300, and inventory holding rate is 25%. Determine

(a) The frequency between reviews if the organization follows a periodic review system.
(b) The order size if the management does not want the inventory level to exceed 5000 units at any time and there are 1500 units currently in stock.
(c) The replenishment policy.

Solution

(a) We have the following information:

- Annual demand, D, is 15,000 units
- Ordering cost, C_o, is \$300 per order
- Inventory holding rate, i, is 15% per year, or 0.15
- Unit cost, C, is \$2.5

Substituting these values in Eq. 3.24, we get

$$T = \sqrt{\frac{2 \times 300}{0.15 \times 15000 \times 2.5}} = 0.3265 \text{ years}$$

This is roughly equal to 119 days or 3 months (assuming 365 days in a year).

(b) If there are 1500 units on hand, and 5000 is the maximum limit on inventory, then the order quantity at this review point would be

$$Q = S - s = 5000 - 1500 = 3500 \text{ units}$$

(c) The replenishment policy, based on period review (T, S) system, can be stated as follows:

Review the inventory status every 119 days. Place one order every 119 days such that the maximum inventory level does not exceed 5000 units.

Solved Problem 3.13
The annual demand for an item is 2000 units, unit cost is \$5, ordering cost is \$25, and inventory holding rate is 30%. Determine the optimal time between reviews if the organization follows a periodic review system

Solution
We have the following information:

- Annual demand, D, is 2000 units
- Ordering cost, C_o, is \$25 per order
- Inventory holding rate, i, is 0.3
- Unit cost, C, is \$5

Substituting these values in Eq. 3.24, we get

$$T = \sqrt{\frac{2 \times 25}{0.3 \times 2000 \times 5}} = 0.129 \text{ years}$$

This is roughly equal to 47 days (assuming 365 days in a year)

Solved Problem 3.14
An inventory manager has the following data with her for an item in her inventory. She would like to know the inventory rate being used if the optimal time between reviews is 30 days (0.082 years, assuming 365 working days a year).

Annual demand for item	: 30,000
Cost per item	: \$5
Ordering cost	: \$100 per order

Solution
From Eq. 3.24, we have

$$i = \frac{2C_o}{T^2 DC}$$

Substituting the values in the above equation, we get

$$i = \frac{2 \times 100}{0.082^2 \times 30000 \times 5} = 0.197$$

The inventory rate being used is 19.7% per annum.

3.6 Summary

In this chapter, we discussed mathematical models to manage inventory of a single item whose demand is known and is constant. We derived an equation for computing EOQ which minimizes the TICs. We learnt that the EOQ model makes several assumptions, and one cannot, therefore, take this model off the shelf and implement it in one's organization. Despite the fact that the model makes several assumptions, the EOQ model is quite popular, possibly because of its robustness and the fact that some of the assumptions made may actually be relaxed.

In this chapter, we briefly discussed extensions of the EOQ model to other situations where supply is not instantaneous (i.e., supply is gradual), and another where planned backlogging (shortages) is allowed. We also discussed computation of the reorder level when lead time is assumed to be constant and known.

As we learned from Chap. 2, there are two types of inventory models (based on timing of review) – continuous review and periodic review. The EOQ model and its extensions come under the continuous review category. Besides discussing the continuous review models, we also discussed a simple periodic review model to determine the optimal time between review intervals.

3.7 Case Study: Fixed Order Quantity System

George Thomas owns a proprietary sports goods manufacturing company – George Sports (GS) – based in Meerut, India. One of the most popular, fast-moving products sold by GS is the cricket bat. GS manufactures different types of cricket bats using local timber. However, one of their products – GSEW7 – is a cricket bat made out of English willow, a type of wood they import from a timber merchant in England.

George has been concerned about the increase in manufacturing costs, and like any other proprietor, he wants to minimize the costs to the extent possible. He calls for Munshi, his trusted accountant for 30 years.

> "I have an important assignment for you Munshi," says George.
>
> "Sure, sir. How can I help?" asks Munshi.
>
> "I am very concerned about the excessive inventory of the GSEW7 logs that I see in the warehouse. I know it is used for one of our fast-moving products but having it in excess does increase the inventory costs and reduces profitability of our company."
>
> "How frequently are we ordering the logs from England?" asks George.
>
> "We don't really have a scientific policy, Sir. I make a visual check of the inventory of the logs and if I feel the stock is less then I place an order," replies Munshi.
>
> "Oh! I thought with a computer in place to manage the stock we could continuously monitor the level of our inventory," exclaims George. "And how many logs do we order every time we place an order?" asks George.
>
> "The order size is not fixed sir. It varies between 150 and 300 logs," replies Munshi.
>
> "Ok. I think this is where the problem is Munshi. We need to come up with an inventory management policy. Please look at your computer records and come back to me with historical inventory data for GSEW7. You have 3 days to get me the data," asserts George.

Back at his office, Munshi starts working on his new assignment of analyzing the demand and cost data for GSEW7. From his computer records, he observes the following:

- Demand for GSEW7 is fairly constant at 200 cricket bats a month.
- Because the wood is imported from England, the ordering cost is high at $800 per order.
- GS uses an inventory carrying and storage rate of 30% per year for accounting purposes.

- The unit cost of each piece of GSEW7 works out to $350.
- The lead time is constant at 30 days.

Case Study Questions

(a) Is the EOQ formula really applicable in this scenario? If so, why? If not, why not?
(b) Considering the data gathered by Munshi, compute the optimal order quantity of English willow logs, assuming one cricket bat of type GSEW7 is made out of one log of English willow.
(c) If GS works 20 days a month, compute the reorder level for English willow logs.
(d) Compute the annual TICs (not including the purchase costs).
(e) Determine the cycle time and the number of orders per year.
(f) The merchant in England can supply English willow logs at a uniform rate of 300 logs each month. If GS works 20 days a month, compute the order quantity for English willow logs as well as the TICs.
(g) Because GS is the sole supplier of English willow bats in the local market, people wanting to buy these specialized bats back-order their requirements in case a bat is out of stock. If the shortage cost is $50, determine the back-ordered quantity and the maximum inventory level.

Answers

(b) Economic order quantity: 191 logs.
(c) Reorder level is 300 logs.
(d) TIC is $20,080.
(e) Cycle time is 19 days, and number of orders per year is 12.5.
(f) If supply is gradual at 300 logs a month, EOQ is 331 and TIC is $11,593.
(g) If shortages are allowed, EOQ is 337 logs, and the maximum inventory level is 109 logs.

3.8 Practice Problems

Problem 3.1
A hardware shop caters to the needs of local manufacturers. One of the fast-moving products that the shop sells is a nylon belt. The monthly demand for nylon belt is 8000 units. The shop orders these belts from its wholesaler and incurs a cost of $35 every time an order is placed. If the holding cost of the nylon belt is $2 per unit per year, compute the following:

(a) Order quantity that would minimize the TICs for the hardware shop
(b) Annual ordering cost
(c) Annual holding cost
(d) TIC

(e) Cycle time
(f) Number of orders in a year

Assume 365 workdays. Also assume back-ordering is not allowed and orders are received in full, instantaneously.

Answers

(a) The EOQ is 1833 units. (Note that in the problem the time units for D and C_h are different).
(b) Annual ordering cost is \$1833.
(c) Annual holding cost is \$1833.
(d) TIC is \$3666.
(e) Cycle time (365 workdays) is 6.97 days.
(f) Number of orders is 52.37 per year.

Problem 3.2
A warehouse stores just one type of item. The annual demand for this item is 1200. The warehouse manager uses a fixed order size of 100 units, equivalent to 1 month's usage, each time she places an order. The inventory carrying rate is 25% per annum, and the cost of the item is \$300. If the ordering cost per order is \$35, compute the cost savings (or losses) if the warehouse manager uses the *EOQ* concept to manage the inventory of this item. Assume no back-ordering.

Answers:

- The annual TICs if the manager orders 100 unit per order is \$4170.
- The annual TICs if the manager uses EOQ is \$2510.
- The total savings (if the manager uses EOQ) would be \$1660 per year.

Problem 3.3
The EOQ for an item is 300 units and its annual demand is 5000 units. If the ordering cost per order is \$20 per order, compute the implied carrying cost for this item. Assume back-orders are not allowed and orders are received in full, instantaneously.

Answer
The implied carrying cost is \$2.22 per unit per year.

Problem 3.4
ScreenShield sells a standard size of window pane laminate. Demand for the laminate is 5000 pieces each year. The inventory holding cost, C_h, is \$2.1 per piece per year, and ordering cost per order, C_o, is \$18 per order. If back-ordering is allowed and the shortage cost is \$5 per piece per year, compute the EOQ, the number of orders, number of back-orders, and the TIC. Also, compute the time over which inventory is on hand and time over which shortages occur.

Answers

- EOQ is 349 units.
- The number of orders is 14.33.
- TIC is: $515.94 (holding cost: $181.67; ordering cost: 257.97; shortage cost: $76.30).
- Maximum number of backorders: 103.2.
- Time over which inventory is on hand: 0.049 years.
- Time over which shortage occurs: 0.020 years.

Problem 3.5
A firm sells an item whose annual demand is 5000 units. If the procurement lead time is a constant 8 days, find the reorder point. Assume 360 workdays a year.

Answer

- Reorder point is 111 units.

Problem 3.6
An accountant of a firm has collated the following inventory data pertaining to a fragile item that costs $50 managed at her firm's warehouse. If the annual demand for the item is 2400, compute the EOQ and the TIC for this item.

Type of cost	Values
Opportunity cost of investment in inventory	8.5%
Fixed cost of order generation per order	$35
Cost of inspecting items received	$5
Cost due to breakage or spoilage	6.5%
Warehouse rental	3%
Insurance costs	1%

Answers:

- Ordering cost is $40 per order.
- Carrying rate is 19% per annum.
- EOQ is 142 units.
- TIC is $1351.

Problem 3.7
You are a consultant for operations of a firm that deals with just one item that costs $45. The firm buys the item wholesale from a supplier and sells retail. You have compiled the following details for the item:

Parameter	Values
Annual demand	4380
Workdays per year	365
Opportunity cost of investment in inventory	12.5%
Fixed cost of order generation per order	$22
Cost of inspecting items received	$3
Cost due to breakage or spoilage	9.5%
Warehouse rental	6.5%
Insurance costs	1.5%

Two options are available for you to analyze:

- Option 1: The supplier can supply all items at once.
- Option 2: The supplier can supply 15 items per day.

Which of the options would you recommend to the firm?

Answers:

Holding rate: 30% annually
Ordering cost: $25 per order

Parameter	All items supplied instantaneously	Items supplied gradually at 15 per day
EOQ	127	284.8
Total holding cost	$860	$384.5
Total ordering cost	$860	$384.5
TICs	$1720	$769

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Chapter 4
Dynamic Inventory Control Models

4.1 Single Price-Break Model

Month over month Rosetta's uses large quantities of vegetable oil for production of food products. Vegetable oil is one of the more expensive ingredients in the production process. To improve profitability, procurement personnel at Rosetta's are always looking at minimizing the costs of this item. If Rosetta's procures vegetable oil from the local supermarket on a need basis, it would cost them an average of $20 per liter. Considering the fact that Rosetta's has been a high-demand loyal customer for several years, the local supplier – Oxxa – informs Rosetta's that if they place an order for 500 liters or more, Oxxa would supply vegetable oil at $19.5 per liter.
Also, on certain occasions Oxxa runs special campaigns to clear off their current stock in anticipation of new, fresh stock of vegetable oil. For a fixed period, they offer a discount of 15% per liter to those buyers that can place an order for larger than usual quantities, before the end of the month.
What should Rosetta's procurement strategy be? Should they continue procuring based on their calculated EOQ, or should they take advantage of the discount and order more? These are the questions that will be addressed in this chapter.

Consider the first part of the running example presented in the box above. The question is should Rosetta's procure 500 liters of vegetable oil because of the $0.5 per liter discount being, or should they order based on their EOQ? This is a single (or one) price-break problem (Vohra 2007). This decision can be made by comparing the TIC for the two scenarios – one without discount at $20 per liter and order size of EOQ, and another at a discounted rate of $19.5 per liter but with an enhanced order size of at least 500 liters.

D. Shenoy, R. Rosas, *Problems & Solutions in Inventory Management*,
DOI 10.1007/978-3-319-65696-0_4

We start by computing the EOQ, using the market price of \$20 per liter. We know the annual demand for vegetable oil is 7200 liters, the ordering cost is \$80 per order, and the inventory holding rate is 30% per year. Using these values, we can determine the EOQ,[1] which is

$$\text{EOQ} = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 20}} = 438 \text{ liters}$$

The TIC[2] in this case would be

$$\text{TIC} = \left(\frac{7200}{438} \times 80\right) + \left(\frac{438}{2} \times (0.3 \times 20)\right) + (7200 \times 20)$$
$$\text{TIC} = 1315 + 1315 + 144000 = \$146,629$$

The next step is to compute the TIC using a discounted rate of \$19.5 per liter. Since this price is available only if the minimum quantity on order is 500 liters, we use this instead of the EOQ value we computed earlier. The TIC in this case is as follows:

$$\text{TIC} = \left(\frac{7200}{500} \times 80\right) + \left(\frac{500}{2} \times (0.3 \times 19.5)\right) + (7200 \times 19.5)$$
$$\text{TIC} = 1152 + 1463 + 140400 = \$143,015$$

Since the TIC for the discounted rate is lower than that of the regular market price, we conclude it is beneficial for Rosetta's to procure 500 liters of vegetable oil. Figure 4.1 illustrates the cost curves for the two scenarios. In this case, a discount of \$0.5 per liter on an order size of 500 liters turned out to be beneficial to Rosetta's. However, this may not always be true as the savings due to purchase price discount may not match the additional carrying costs that may be incurred, in which case it is prudent to maintain an order size that equals EOQ.

Solved Problem 4.1

A manufacturer purchases 1200 units of an item from a supplier every year. The ordering cost is \$250 per order, inventory rate is 15% per year, and the cost of the item is \$100. Based on this information compute the following:

(a) Compute the economic order quantity assuming no shortages are allowed and no discount is being offered by the supplier.

---

[1] From Eq. 3.8 in Chap. 3.

[2] We include the purchase price in calculation of the TIC for this class of problems because discounts impact the total purchase costs.

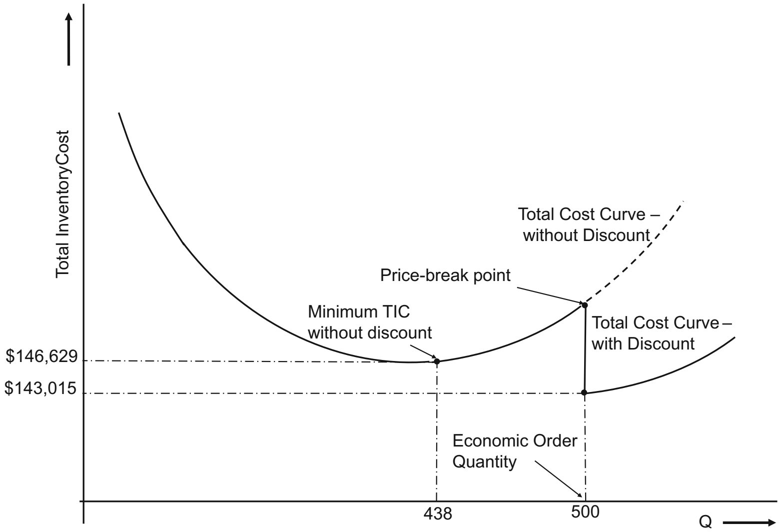

Fig. 4.1 Cost curve – single price-break model

(b) The supplier offers a discount of 2% if the manufacturer places an order of not less than 750 units each time they order. Should the manufacturer accept the discounted price?

(c) What would be the minimum acceptable discounted price if the manufacturer uses an order size of 750 units?

Solution

(a) The EOQ assuming no shortages are allowed and no discount is being offered can be determined using Eq. 3.8. Substituting the values, we get

$$\text{EOQ} = \sqrt{\frac{2 \times 1200 \times 250}{0.15 \times 100}} = 200 \text{ units}$$

The TIC[3] in this case is

$$\text{TIC}_{100} = \left(\frac{1200}{200} \times 250\right) + \left(\frac{200}{2} \times (0.15 \times 100)\right) + (1200 \times 100)$$

or

$$\text{TIC}_{100} = 1500 + 1500 + 120000 = \$123,000$$

[3]We will use the purchase price as subscripts in this chapter.

(b) The TIC if the supplier offers a discount of 2% (or a purchase price or \$98 per unit) for an order size not less than 750 units is

$$\mathrm{TIC}_{98} = \left(\frac{1200}{750} \times 250\right) + \left(\frac{750}{2} \times (0.15 \times 98)\right) + (1200 \times 98)$$

or

$$\mathrm{TIC}_{98} = 400 + 5513 + 117600 = \$123,513$$

Notice that $\mathrm{TIC}_{98} > \mathrm{TIC}_{100}$. Therefore, the manufacturer must not accept the discounted price for an order size of 750 units.

(c) We assume that the minimum acceptable discount price for order size of 750 units is k. The TIC in this case would be

$$\begin{aligned}\mathrm{TIC}_{100-k} = &\left(\frac{1200}{750} \times 250\right) + \left(\frac{750}{2} \times \left(0.15 \times \frac{100-k}{100} \times 100\right)\right) \\ &+ \left(1200 \times \frac{100-k}{100} \times 100\right) \\ = &\ 123000\end{aligned}$$

or

$$= 400 + 5625\left(\frac{100-k}{100}\right) + 120000\left(\frac{100-k}{100}\right) = 123000$$

Simplifying, we get

$$125625\left(\frac{100-k}{100}\right) = 122600$$

or

$$\left(\frac{100-k}{100}\right) = 0.9759$$

or $k = 2.41\%$. The minimum acceptable discount price for order size of 750 units is 2.41% (or purchase price must be \$97.59 per unit).

In the following sections, we will discuss models with multiple price-breaks.

4.2 All-Units Discount: Instantaneous Supply Model

Consider a new business deal between Rosetta's and Oxxa shown below:

Business Deal – Oxxa and Rosetta's
The market price of vegetable oil is \$20 per liter. If Rosetta's procures 450 liters (or more) of vegetable oil each time they place an order, Oxxa will offer them a discount of 10% over the prevailing market rate. The discount rate would increase to 20% if Rosetta's procures 500 liters or more per order. Further, Oxxa will supply the ordered quantities immediately.

Table 4.1 shows the discount and effective price offered by Oxxa for different ranges of quantities of vegetable oil. This is referred to as multiple price-break (discount) schedule. It should be noted that the assumptions we used in derivation of the Basic EOQ model in Chap. 3 will continue to apply in this scenario as well.

Let us now use the information provided in Table 4.1 and determine the optimal and feasible order quantity under discount. This can be achieved in the following four steps:

Step 1: Compute Order Size for All Values of Purchase Price
The first step is to compute the EOQ for each of the price-break values of $\$20$, $\$18$, and $\$16$. We will use the following data that have been used in Chap. 3:

- Annual demand for vegetable oil is 7200 liters.
- Ordering cost C_o is $\$80$ per order.
- Unit cost C is $\$20$ per liter (market price), \$18 per liter if the order size is more than 450 liters, \$16 per liter if the order size is more than 500 liters.
- Inventory holding rate i is 30%.

Using the above data, we can compute the EOQ values for each of the price-break points, which are as follows:

$$\text{EOQ}_{20} = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 20}} = 438 \text{ liters}$$

Similarly,

$$\text{EOQ}_{18} = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 18}} = 462 \text{ liters}$$

Table 4.1 Discount schedule

Order quantity range	Discount rate (Offer price)	Effective purchase price per liter
0–450 liters	Nil	\$20
450–500 liters	10%	\$18
More than 500 liters	20%	\$16

Table 4.2 Feasibility check

Purchase price	Order quantity	Feasible	Adjusted EOQ
$20	438	Yes	–
$18	462	Yes	–
$16	490	No	500

and

$$\mathrm{EOQ}_{16} = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 16}} = 490 \text{ liters}$$

Step 2: Check Feasibility of Order Quantities

Next, we analyze the order quantities we computed. We see that EOQ_{16} is infeasible since it does not fall in the range of 500+ liters. In other words, a discounted rate of $16 is offered only if the order size is more than 500 liters; however, the EOQ we computed for purchase price of $16 is less than 500 liters, and hence it is considered infeasible. We, therefore, adjust the minimum value of EOQ_{16} to 500 liters (from 490 liters). Table 4.2 summarizes the feasibility of the order quantities for each purchase price as well as the adjusted order quantity (adjusted EOQ).

The EOQs for purchase price of $20 and $18 are in the feasible range, and no adjustment is required.

Step 3: Determine TIC

We next compute TIC for EOQ_{20}, EOQ_{18}, and EOQ_{16} using the following equation[4]:

$$\mathrm{TIC} = \frac{D}{Q} C_o + \frac{Q}{2} iC + DC \tag{4.1}$$

Substituting the values in Eq. 4.1, we get

$$\mathrm{TIC}_{20} = \left(\frac{7200}{438} \times 80\right) + \left(\frac{438}{2} \times (0.3 \times 20)\right) + (7200 \times 20)$$

$$\mathrm{TIC}_{20} = 1315 + 1315 + 144000 = \$146,629$$

Similarly,

$$\mathrm{TIC}_{18} = \left(\frac{7200}{462} \times 80\right) + \left(\frac{462}{2} \times (0.3 \times 18)\right) + (7200 \times 18)$$

$$\mathrm{TIC}_{18} = 1247 + 1247 + 129600 = \$132,094$$

and

[4] This equation was derived in Chap. 3.

Table 4.3 Summary of Q and TIC

Unit cost	Order quantity	TIC
\$20	438	146,629
\$18	462	132,094
\$16	500	117,552

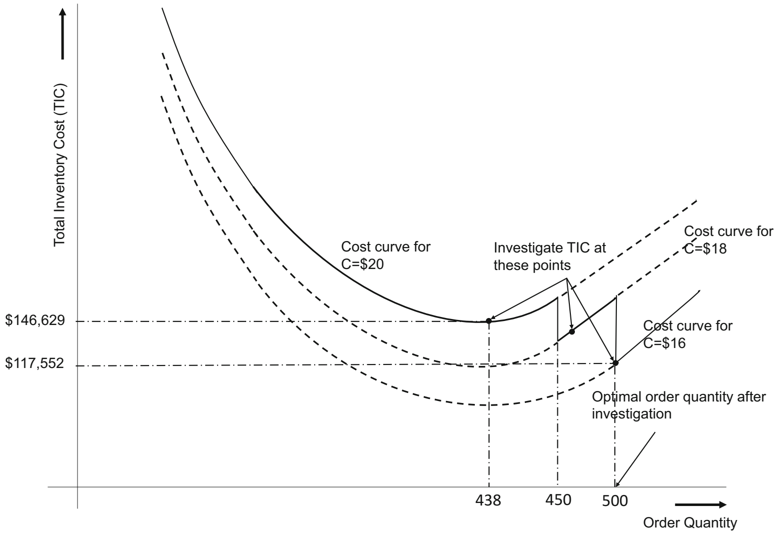

Fig. 4.2 Multiple price-break model

$$\text{TIC}_{16} = \left(\frac{7200}{500} \times 80\right) + \left(\frac{500}{2} \times (0.3 \times 16)\right) + (7200 \times 16)$$

$$\text{TIC}_{16} = 1152 + 1200 + 115200 = \$117,552$$

Step 4: Determine Optimal Order Size

Table 4.3 summarizes the results of this problem including TIC values for each option.

Notice that the TIC_{16} is less than both TIC_{20} and TIC_{18}. The order quantity corresponding to TIC_{16} is 500 liters. Thus, we conclude that it is best to order 500 liters of vegetable oil each time we place an order. Figure 4.2 illustrates the price curves and optimal order quantity for this multiple price-break model.

4.3 Summary of All-Units Discount (Instantaneous Supply) Solution Procedure

Following is a summary of the all-units discount solution procedure (Nahmias 2005):

- For each unit cost value, use the basic EOQ formula to compute the economic order quantity.

- If the EOQ value computed falls in the feasible range, compute the TIC. If the EOQ does not fall in the feasible range, adjust the EOQ such that it falls in the feasible range. Compute the TIC using this adjusted EOQ value.
- Compare the TIC values for all such feasible EOQ values. The EOQ value which corresponds to the minimum TIC is the optimal quality under all-unit discount.

The following section illustrates the application of the all-units discount theory.

Solved Problem 4.2
The annual consumption of sugar used in a bakery is 10,400 kg, the carrying cost is 20% of the average inventory valuation, and the ordering cost is $200 per order. If the sugar supplier offers the bakery quantity discounts as shown in Table 4.4, use the concept of optimal order quantity to determine the EOQ strategy that best suits the needs of the bakery.

Solution:
The solution for an inventory problem with quantity discounts can be found in four steps:

Step 1: Compute Order Size for All Values of Purchase Price
Using the EOQ formula, we first determine the order quantities for each purchase price option. The calculations are shown in Table 4.5.

Step 2: Check Feasibility of Order Quantities
In this step, we validate the calculated EOQs against the quantity range. If the calculated EOQ does not fall in the range of quantity discount, we adjust the EOQ to the minimum in the quantity range. Table 4.6 summarizes the feasibility of the order quantities for each purchase price, as well as the adjusted order quantity.

Table 4.4 Discount schedule – Solved Problem 4.2

Quantity range	Unit price
0–1499 kg	$10
1500–5000	$9.90
5000+ kg	$9.80

Table 4.5 EOQ calculation for each price option

EOQ @ $10	EOQ @ $9.9	EOQ @ $9.8
$= \sqrt{\dfrac{2 \times 10400 \times 200}{10 \times 0.2}}$ $= 1442$ units	$= \sqrt{\dfrac{2 \times 10400 \times 200}{9.9 \times 0.2}}$ $= 1449$ units	$= \sqrt{\dfrac{2 \times 10400 \times 200}{9.8 \times 0.2}}$ $= 1457$ units

Table 4.6 Feasibility check – Solved Problem 4.2

Purchase price	Order quantity	Feasible	Adjusted EOQ
$10	1442	Yes	–
$9.9	1449	No	1500
$9.8	1457	No	5000

Table 4.7 Summary of TIC – Solved Problem 4.2

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
EOQ @ 10	1442	1442	1442	104,000	$10,6884
EOQ @ 9.9	1500	1387	1485	102,960	$10,5831
EOQ @ 9.8	5000	416	4900	101,920	$10,7236

Table 4.8 EOQ calculations for each source

EOQ @ $0.65 (AJW)	EOQ @ $0.60 (BJW)	EOQ @ $0.55 (CJW)
$= \sqrt{\frac{2 \times 20000 \times 25}{0.65 \times 0.15}}$ $= 3203$ notebooks	$= \sqrt{\frac{2 \times 20000 \times 25}{0.60 \times 0.15}}$ $= 3333$ notebooks	$= \sqrt{\frac{2 \times 20000 \times 25}{0.55 \times 0.15}}$ $= 3481$ notebooks

Step 3: Determine TIC Using Adjusted EOQ
TIC can be determined by using Eq. 4.1. The TIC for the three strategies are as shown in Table 4.7.

Step 4: Determine Best Strategy
Since the minimum TIC corresponds to the EOQ strategy of $9.9, we conclude that the best strategy is to order 1500 kg of sugar each time an order is placed, at a purchase price of $9.9 per kg.

Solved Problem 4.3
You own a stationery outlet in a college where you sell one type of student notebooks. You purchase your stock of student notebooks from one of the local wholesalers and sell it to students in the college for $0.85 per notebook. One wholesaler – AJW – sells you notebooks at $0.65, irrespective of the quantity of notebooks you purchase from them. Another wholesaler – BJW – sells notebooks to you at 0.60 provided you place an order for at least 3500 notebooks. Yet another source – CJW – sells the same notebooks at 0.55, but the minimum lot size must be 4500 or above. If the annual demand for student notebooks is 20,000 and the ordering cost per order is $25, which wholesaler would you procure your stock from? Assume an annual rate of interest of 15% for calculating your carrying costs.

Solution:

Step 1: Compute Order Size for All Options
Using the EOQ formula, we can determine the order quantities for each purchase price option (in this case, the procurement source). Calculations are shown in Table 4.8.

Step 2: Check Feasibility
Table 4.9 shows the feasibility check for each of the strategies. Since the EOQs computed for BJW and CJW are not within the discount range, we adjust those EOQs to the minimum quantity where discounts would be available.

Table 4.9 Feasibility check – Solved Problem 4.3

Purchase price	Order quantity	Feasible	Adjusted EOQ
$0.65	3203	Yes	–
$0.60	3333	No	3500
$0.55	3481	No	4500

Table 4.10 Summary of TIC – Solved Problem 4.3

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
AJW@0.65	3203	156	156	13,000	$13,312
BJW@0.60	3500	142	158	12,000	$12,300
CJW@0.55	4500	111	186	11,000	$11,297

Step 3: Determine TIC Using Adjusted EOQ
TIC can be determined by using Eq. 4.1. The TIC for the three strategies are as shown in Table 4.10.

Step 4: Determine Best Strategy
From Table 4.10, we notice that the minimum TIC corresponds to CJW. The best strategy is, therefore, to order 4500 notebooks each time an order is placed, at a purchase price of $0.55 per notebook.

4.4 All-Units Discount: Gradual Supply Model

In the previous section, we assumed a scenario that orders would be filled instantaneously, in one lot. Let us now consider a different scenario – that of gradual supply. The procedure to determine the order size is similar to that in the all-units discount (instantaneous supply) case, with the only difference being in the formulae we use to compute the order size and the total inventory costs (Gaither 1987). The formulae used to compute the EOQ and TIC for the gradual supply case are as follows[5]:

$$Q = \sqrt{\frac{2DC_o}{iC}\left(\frac{p}{p-d}\right)} \tag{4.2}$$

and

[5]These were derived in Chap. 3.

Table 4.11 Discount schedule – Solved Problem 4.4

Quantity range	Unit cost
0–1499 kg	\$10
1500–5000	\$9.90
5000+ kg	\$9.80

Table 4.12 Feasibility check – Solved Problem 4.4

Purchase price	Order quantity	Feasible	Adjusted EOQ
\$10	4079	No	NA
\$9.9	4100	Yes	4100
\$9.8	4121	No	5000

$$\text{TIC} = \frac{Q}{2}\left(\frac{p}{p-d}\right)iC + \frac{D}{Q}C_o + DC \tag{4.3}$$

The theory and concept of the all-units discount – gradual supply model is illustrated with a numerical example.

Solved Problem 4.4

The annual consumption of sugar used in a bakery is 10,400 kg, the carrying cost is 20% of the average inventory valuation, and the ordering cost is \$200 per order. If the supplier offers the bakery a quantity discount as shown in Table 4.11, use the concept of optimal order quantity to determine the EOQ strategy that best suits the needs of the bakery if the daily requirement is 35 kg per day while the supplier can supply at a uniform rate of 40 kg per day.

Solution:

Step 1: Compute Order Size for All Options

The first step is to compute the EOQ for all price options, using Eq. 4.2. The computations are shown below:

$$Q_{10} = \sqrt{\frac{2 \times 10400 \times 200}{10 \times 0.20} \times \left(\frac{40}{40-35}\right)} = 4079 \text{ kg}$$

$$Q_{9.9} = \sqrt{\frac{2 \times 10400 \times 200}{9.9 \times 0.20} \times \left(\frac{40}{40-35}\right)} = 4100 \text{ kg}$$

$$Q_{9.8} = \sqrt{\frac{2 \times 10400 \times 200}{9.8 \times 0.20} \times \left(\frac{40}{40-35}\right)} = 4121 \text{ kg}$$

Step 2: Check Feasibility

The next step is to check the feasibility of the computed EOQs. Table 4.12 summarizes the feasibility check. As can be seen from the table, the EOQ strategy for price of \$9.8 is infeasible. We therefore adjust the EOQ upward to 5000 kg. The EOQ strategy for \$10 is not within the feasible range, and there is no way we can adjust the EOQ for that price strategy.

Table 4.13 Summary of TIC – Solved Problem 4.4

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
$9.9	4100	507	507	102,960	$102,960
$9.8	5000	416	612	101,920	$102,949

Step 3: Determine TIC Using Adjusted EOQ
TIC can be determined by using Eq. 4.3. The TICs for the two feasible strategies are as shown in Table 4.13.

Step 4: Determine Best Strategy
As can be seen from Table 4.13, the minimum TIC corresponds to an EOQ strategy with a price of $9.8 per kg. The best strategy is, therefore, to order 5000 kg each time an order is placed.

4.5 Incremental Discount Model

In the previous section, we learnt about application of theory to situations where discount is offered on all units. Realistically, wholesalers may not offer a discounted price on all units purchased. They may offer no discount on the first few units purchased, a small discount on the next few units purchased, and a larger discount if the purchase exceeds a preset threshold of units purchased. This discount model is called incremental discount (Srinivasan 2010). Let us now study the solution procedure for this model. Consider a scenario where Oxxa offers discount to Rosetta's as follows:

Business Deal – Oxxa and Rosetta's
Each liter of vegetable oil will cost Rosetta's $20 per liter for quantities up to 450 liters. For an order size of 450 liters and up to 500 liters, Oxxa will offer Rosetta's a discount of 10% over the prevailing market rate. The discount rate would increase to 20% if Rosetta's procures 500 liters or more per order.

In the all-units discount case, Rosetta's would have been offered a discount of 20% on the entire order, if the order size exceeds 500 liters. In the incremental discounts scenario, Rosetta's will have to pay the market price for the first 450 liters. A discount of 10% will be offered on the 450th liter onward up to 500 liters. A discount of 20% would be offered from the 500th liter onward. This is the key difference between all-units discount and incremental discount model. In this case, the purchase price reduces incrementally as additional quantity is procured.

Mathematically,

- Let m be the number of price bands and $q_1, q_2, q_3, \ldots, q_m$ be the price break quantities where $q_1 = 0$;
- Let us also assume that the unit purchasing cost is C_j;
- Let Q represent the optimal quantity that we plan to procure, and let us further assume that this quantity falls in the j^{th} price band, where $j \in m$.

Under these assumptions, the purchasing cost $C(Q)$ for a quantity Q can be given by

$$C(Q) = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \ldots + C_{j-1}\left(q_j - q_{j-1}\right) + C_j\left(Q - q_j\right)$$

Note that only the extreme right term is a function of Q. Therefore, we can rewrite $C(Q)$ as

$$C(Q) = S_j + C_j\left(Q - q_j\right)$$

where S_j is the sum of other non-Qcontaining terms given by

$$S_j = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \ldots + C_{j-1}\left(q_j - q_{j-1}\right)$$

The average purchasing price is therefore

$$\frac{C(Q)}{Q} = \frac{S_j}{Q} + \frac{C_j\left(Q - q_j\right)}{Q}$$

The inventory holding cost is a function of the average purchasing price and the inventory holding rate i. Thus, the holding cost function is

$$= \frac{Q}{2} \times \frac{C(Q)}{Q} i$$

The ordering cost is a function of the number of orders and the cost per order. This can be expressed as

$$= \frac{D}{Q} C_o$$

The investment in inventory is a function of the demand and the average purchase price. This is therefore

$$= \frac{DC(Q)}{Q}$$

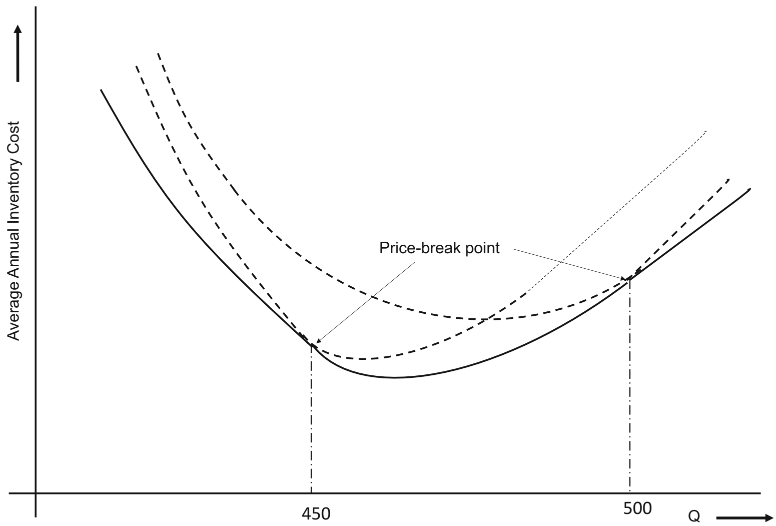

Fig. 4.3 Incremental discount – Cost function

Table 4.14 Discount schedule for incremental discount problem

Order quantity range	Discount rate (Offer price)	Effective purchase price per unit
0–450 liters	Nil	\$20
450–500 liters	10%	\$18
More than 500 liters	20%	\$16

The total inventory cost for the incremental discount model can be expressed as

$$\mathrm{TIC} = \frac{D}{Q} C_o + \frac{QC(Q)}{2Q} i + \frac{DC(Q)}{Q} \tag{4.4}$$

The value of Q which is feasible, and which results in the minimum TIC, is considered to be optimal. Figure 4.3 shows the incremental discount cost function.

Let us now use this theory to compute the optimal order quantity for the Rosetta's–Oxxa example, assuming Oxxa offers an incremental discount. We have the following information with us:

- Annual demand is 7200 liters
- Ordering cost C_o is \$80 per order
- Inventory holding rate i is 30%

In addition, we also know that Oxxa offers an incremental discount, as shown in Table 4.14.

If Q is the order quantity, mathematically, the above can be expressed as follows:

$$C(Q) = \begin{cases} 20Q \text{ for } 0 \leq Q < 450 \\ 9000 + 18(Q - 450) \text{ for } 450 \leq Q < 500 \\ 9900 + 16(Q - 500) \text{ for } 500 \leq Q \end{cases}$$

Dividing the above by Q, we get

$$\frac{C(Q)}{Q} = \begin{cases} 20 \text{ for } 0 \leq Q < 450 \\ \frac{900}{Q} + 18 \quad \text{for } 450 \leq Q < 500 \\ \frac{1900}{Q} + 16 \quad \text{for } 500 \leq Q \end{cases}$$

The average purchase price of vegetable oil can be determined by Eq. 4.4. Let us now compute the TIC for each value of $\frac{C(Q)}{Q}$.

Step 1: Compute Order Size for each Purchase Price Option

Case A: $\frac{C(Q)}{Q} = 20$ Using Eq. 4.4, and substituting the values, we get

$$TIC_{20} = \left(\frac{7200 \times 80}{Q}\right) + \left(\frac{Q}{2} \times 20 \times 0.3\right) + (7200 \times 20)$$

To obtain the minima for Q, we differentiate TIC function w.r.t Q and equate it to 0. We get

$$Q_{20} = \sqrt{\frac{2 \times 7200 \times 80}{20 \times 0.3}} = 438$$

Case B: $\frac{C(Q)}{Q} = \frac{900}{Q} + 18$

Substituting the values in Eq. 4.4, we get

$$TIC_{18} = \left(\frac{7200 \times 80}{Q}\right) + \left(\frac{Q}{2} \times \left(\frac{900}{Q} + 18\right) \times 0.3\right) + \left(7200 \times \left(\frac{900}{Q} + 18\right)\right)$$

$$Q_{18} = \sqrt{\frac{2 \times 7200 \times (80 + 900)}{18 \times 0.3}} = 1617$$

Table 4.15 Feasibility check – Incremental discount

Purchase price	Order quantity	Feasible?
$20	438	Yes
$18	1617	No
$16	2437	Yes

Case C: $\frac{C(Q)}{Q} = \frac{1900}{Q} + 16$

Substituting the values in Eq. 4.4, we get

$$\mathrm{TIC}_{16} = \left(\frac{7200 \times 80}{Q}\right) + \left(\frac{Q}{2} \times \left(\frac{1900}{Q} + 16\right) \times 0.3\right) + \left(7200 \times \left(\frac{1900}{Q} + 16\right)\right)$$

$$Q_{16} = \sqrt{\frac{2 \times 7200 \times (80 + 1900)}{16 \times 0.3}} = 2437$$

Step 2: Check Feasibility of Order Quantities

The next step is to check if the estimated quantities Q are all in the feasible range. Table 4.15 summarizes the feasibility check for each option.

In this case, we notice that Q_{18} must be in the range of 450–500 liters to obtain a discount of 10%. However, the estimated Q_{18} value is outside that range. Therefore, we consider Q_{18} to be infeasible. Other options are feasible.

Step 3: Compute TICs

The next step is to compute the TICs using Eq. 4.4. Using the values for Q_{20} and Q_{16} obtained earlier, we can compute the TIC as follows:

$$\mathrm{TIC}_{20} = \left(\frac{7200 \times 80}{438}\right) + \left(\frac{438}{2} \times 20 \times 0.3\right) + (7200 \times 20)$$

$$\mathrm{TIC}_{20} = 1315 + 1314 + 144000 = \$146,629$$

Similarly,

$$\mathrm{TIC}_{16} = \left(\frac{7200 \times 80}{2437}\right) + \left(\frac{2437}{2} \times \left(\frac{1900}{2437} + 16\right) \times 0.3\right) + \left(7200 \times \left(\frac{1900}{2437} + 16\right)\right)$$

$$\mathrm{TIC}_{16} = 236 + 6133 + 120813 = \$127,182$$

Step 4: Determine the Best Strategy

The final step is to compare the TIC values. We see that the $\mathrm{TIC}_{16}$is less than TIC_{20}. Therefore, it is best to order 2437 liters of vegetable oil each time we place an order.

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
\$20	438	1315	1314	144,000	\$146,629
\$16	2437	236	6121	120,813	\$127,182

4.6 Summary of Incremental Discount Solutions Procedure

Following is a summary of the incremental discount solution procedure:

- For each price band, formulate the cost function $\frac{C(Q)}{Q}$.
- Compute EOQ for each price option (case). Compute the TIC for the EOQs that are feasible.
- Compare the TIC values for all such feasible EOQs. The EOQ which corresponds to the minimum TIC is the optimal quality.

4.7 All-Units Discount and Incremental Discount: A Comparison

Let us compare the solutions presented by using the all-units discount and the incremental discount strategies. Table 4.16 shows the EOQ and the TIC values obtained for the feasible solution using both the methods. We see that while the minimum cost obtained using all-units discount methods occurs at the price-break point (500 units), whereas the minima for the incremental discount does not occur at the price-break quantity (2437 units).

Following is another solved problem that illustrates the application of the incremental discount model.

Solved Problem 4.5
A manufacturer of mircoturbines sources plastic impellers from its trusted supplier. Every time the manufacturer places an order for procuring impellers, it incurs a cost of \$20. The carrying cost rate is 20%. The unit purchase price is based on the size of the order as shown in Table 4.17. If the manufacturer has an annual demand for 1000 turbines, compute the optimal order quantity if the supplier uses an incremental discount schedule.

If Q is the order quantity, mathematically, the above can be written as

$$C(Q) = \begin{cases} 3Q \text{ for } 0 \leq Q < 300 \\ 900 + 2.9(Q - 300) \text{ for } 300 \leq Q < 600 \\ 1770 + 2.8(Q - 600) \text{ for } Q < 600 \end{cases}$$

Table 4.16 Comparison of discount strategies

All-units discount			Incremental discount		
Strategy	EOQ	TIC	Strategy	EOQ	TIC
\$16	500	\$117,552	\$16	2437	\$127,182

Table 4.17 Discount schedule – Solved Problem 4.5

Order quantity range	Effective cost per unit
0–300 items	\$3.00
300–600 items	\$2.90
More than 600 items	\$2.80

Dividing by Q throughout and simplifying, we can reduce the above set of equations to

$$\frac{C(Q)}{Q} = \begin{cases} 3 \text{ for } 0 \le Q < 300 \\ 2.9 + \dfrac{30}{Q} \text{ for } 300 \le Q < 600 \\ 2.8 + \dfrac{90}{Q} \text{ for } Q < 600 \end{cases}$$

The average purchase price of the item can be determined by the cost function given below:

$$\text{TIC} = \frac{D}{Q}C_o + \frac{QC(Q)}{2Q}i + \frac{DC(Q)}{Q}$$

Let us now compute the TIC for each value of $\frac{C(Q)}{Q}$.

Step 1: Compute Order Size for each Purchase Price Option

Case A: $\frac{C(Q)}{Q} = 3$ Using the above equation and substituting the values, we get

$$\text{TIC}_3 = \left(\frac{1000 \times 20}{Q}\right) + \left(\frac{Q}{2} \times 3 \times 0.2\right) + (1000 \times 3)$$

To obtain the minima for Q, we differentiate TIC function w.r.t Q and equate it to 0. We get

$$Q_3 = \sqrt{\frac{2 \times 1000 \times 20}{3 \times 0.2}} = 258$$

Case B: $\frac{C(Q)}{Q} = \frac{30}{Q} + 2.9$

Using the above equation and substituting the values, we get

$$\text{TIC}_{2.9} = \left(\frac{1000 \times 20}{Q}\right) + \left(\frac{Q}{2} \times \left(\frac{30}{Q} + 2.9\right) \times 0.2\right) + \left(1000 \times \left(\frac{30}{Q} + 2.9\right)\right)$$

Table 4.18 Feasibility check – Solved Problem 4.5

Purchase price	Order quantity	Feasible?
\$3.0	258	Yes
\$2.9	415	Yes
\$2.8	626	Yes

To obtain the minima for Q, we differentiate the TIC function w.r.t Q and equate it to 0. We get

$$Q_{2.9} = \sqrt{\frac{2 \times 1000 \times (20 + 30)}{2.9 \times 0.2}} = 415$$

Case C: $\frac{C(Q)}{Q} = \frac{90}{Q} + 2.8$

Using the above equation and substituting the values, we get

$$\mathrm{TIC}_{2.8} = \left(\frac{1000 \times 20}{Q}\right) + \left(\frac{Q}{2} \times \left(\frac{90}{Q} + 2.8\right) \times 0.2\right) + \left(1000 \times \left(\frac{90}{Q} + 2.8\right)\right)$$

$$Q_{2.8} = \sqrt{\frac{2 \times 1000 \times (20 + 90)}{2.8 \times 0.2}} = 626$$

Step 2: Check Feasibility of Order Quantities

The next step is to check if the estimated quantities Q are all in the feasible range. Table 4.18 summarizes the feasibility check for each price strategy.

In this case, we notice that all EOQ values are feasible. So we proceed to compute the TIC values.

Step 3: Compute TICs

The next step is to compute the TICs, which is given by

$$\mathrm{TIC} = \frac{D}{Q} C_o + \frac{QC(Q)}{2Q} i + \frac{DC(Q)}{Q}$$

Using the values for Q_3 obtained earlier, we can compute the TIC as follows:

$$\mathrm{TIC}_3 = \left(\frac{1000 \times 20}{258}\right) + \left(\frac{258}{2} \times 3 \times 0.2\right) + (1000 \times 3)$$

$$\mathrm{TIC}_3 = 77.51 + 77.4 + 3000 = \$3,154.91$$

Similarly,

$$\mathrm{TIC}_{2.9} = \left(\frac{1000 \times 20}{415}\right) + \left(\frac{415}{2} \times \left(\frac{30}{415} + 2.9\right) \times 0.2\right) + \left(1000 \times \left(\frac{30}{415} + 2.9\right)\right)$$

Table 4.19 Determine the best strategy – Solved Problem 4.5

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
$3.0	258	77.51	77.40	3000	$3154.91
$2.9	415	48.19	123.35	2972.28	$3143.83
$2.8	626	31.94	184.28	2943.76	$3159.98

$$\mathrm{TIC}_{2.9} = 48.19 + 123.35 + 2972.28 = \$3,143.83$$

and

$$\mathrm{TIC}_{2.8} = \left(\frac{1000 \times 20}{626}\right) + \left(\frac{626}{2} \times \left(\frac{90}{626} + 2.8\right) \times 0.2\right) + \left(1000 \times \left(\frac{90}{626} + 2.8\right)\right)$$

$$\mathrm{TIC}_{2.8} = 31.94 + 184.28 + 2943.76 = \$3,159.98$$

Step 4: Determine the Best Strategy
The final step is to determine the best strategy by comparing the TIC values. Table 4.19 summarizes the TIC values for all the EOQ strategies. We notice that the minimum TIC corresponds to a price strategy of $2.9 per unit. The optimal order quantity is therefore 415 units.

4.8 One-Off, Fixed-Period Discount (Special Discount)

In certain situations, suppliers may offer a special, one-off discount. This is usually done by suppliers to sell off existing stock of goods before the arrival of fresh stock (Chopra and Meindl 2010).

Consider a scenario where Oxxa has a large amount of vegetable oil on their stock and are expecting a fresh stock next month. They would like to clear off the existing stock of vegetable oil before the arrival of fresh stock So, they run a campaign to offer a one-time, fixed period discount of 15% per liter to those buyers, usually retailers, that can place an order for larger than usual quantities, before the end of the month. Retailers would normally order vegetable oil based on their computation of *EOQ* which is given by

$$\mathrm{EOQ} = \sqrt{\frac{2DC_o}{iC}}$$

However, when a wholesaler offers discounts, retailers may order more than EOQ to reduce their future cost of sales. The optimal order quantity at discounted price, Q_d, is given by

$$Q_d = \frac{dD}{(C-d)i} + \frac{CQ}{(C-d)} \tag{4.5}$$

where d is the discount value. This model is valid under the following assumptions:

- The discount is one-time, for a fixed period.
- This discount may not be offered in future.
- The customer demand (from retailer's point of view) remains constant.
- The item is not perishable.
- The period over which the demand is analyzed is an integer of the original optimal order quantity, Q

Let us now use this concept to solve Rosetta's order quantity problem under one-time discount. We will use the same data that we have used earlier (reproduced below):

- Annual demand is 7200 liters
- Ordering cost C_o is $\$80$ per order
- Inventory holding rate i is 30%

Original (before discount) purchase price is $20 per liter. Using the above data, we first determine the optimal order quantity, Q, which is

$$\text{EOQ} = \sqrt{\frac{2 \times 7200 \times 80}{0.3 \times 20}} = 438 \text{ liters}$$

Next, we determine the optimal order quantity under one-time discount. A 15% discount on $20 purchase price works out to $3 per liter. Using the data available and substituting in Eq. 4.5, we get

$$Q_d = \frac{3 \times 7200}{(20-3) \times 0.3} + \frac{20 \times 438}{(20-3)} = 4236 + 515 = 4751 \text{ liters}$$

Observe that a 15% discount in price increases the optimal order quantity by approximately 11 times (or 984%) (Table 4.20).

Since the one-off discount increases the replenishment order size, the inventory is expected to stay on shelf for a longer period of time. This concept needs to be applied with care to items that have limited life or those that become obsolete faster.

Table 4.20 Sensitivity of EOQ to One-off discount

Price	EOQ (liters)
$20 (market price)	438
$17 (discount of 15%, one-time)	4751

Solved Problem 4.6
Binifont is a retail chain that sells PuraHoney, a popular brand of honey manufactured by PH Foods. The annual demand for PuraHoney experienced by Binifont in the local market is 1200 jars. PH Foods charges \$5 per jar, and Binifont uses an inventory holding rate of 25% per annum.

PH Foods has just announced a one-time promotional discount of \$1 on all jars of honey procured by retailers over the next 15 days. Compute the optimal order quantity that Binifont needs to procure during the promotional period if they incur an ordering cost of \$30 per order.

Solution
The first step is to compute EOQ, which is given by

$$\text{EOQ} = \sqrt{\frac{2 \times 1200 \times 30}{0.25 \times 5}} = 240 \text{ jars}$$

Next, we use Eq. 4.5 to compute the optimal order quantity under one-off discount

$$Q_d = \frac{1 \times 1200}{(5 - 1) \times 0.25} + \frac{5 \times 240}{(5 - 1)} = 1200 + 300 = 1500 \text{ jars}$$

During the promotion, Binifont needs to place an order for 1500 jars of honey.

4.9 Summary

When a seller offers a discount, the buyer must decide on the size of the purchase order that would minimize the TICs. The buyer will have to balance the reduced purchase cost with the increase in carrying cost. In this chapter, we discussed continuous review (EOQ) price-break models. Besides the single price model, we discussed the following three types of multiple price-break models in this chapter:

- All-units discount model, where the supplier offers a uniform discount on all the units purchased. Instantaneous and gradual supply models were both discussed.
- Incremental discount model, where additional units purchased are offered a higher discount.
- Special discount model, also known as one-off discount or promotional discount model.

A key takeaway from this chapter is that the minimum cost obtained using all-units discount method occurs at the price-break point, whereas the minima for the incremental discount does not occur at the price-break quantity. Another takeaway (one-off discount model) is that a small discount in the price increases the size of the optimal order quantity several times. Because of this, one needs to apply caution before the one-off discount model can be applied to items that have limited shelf life.

4.10 Case Study: All-Units Discount

Yano is an electronic toy manufacturing company, based in Melbourne, Victoria. Yano manufactures a variety of electronic toys that "speak" to their patrons – children. Over the last 15 years, people at Yano have mastered the art of creating and packaging toys.

A lot of electronics and circuitry goes into the making of a "speaking electronic" toy. One key component of any electronic toy built by Yano is a circuit board. In the past, Yano have sourced their circuit boards from SK Electronics, also based in Melbourne. Yano incurs a cost of $300 each time they place an order. SK Electronics have been very reliable and have supplied the required number of circuit boards to Yano. Circuit boards supplied are stored in Yano's stock control room. The inventory manager at Yano uses an annual inventory carrying rate of 35%.

Felix is a Product Manager at Yano. He has been with Yano for the last 15 years and is one of the architects of Yano. One day, Felix receives a meeting request from Tim, the Chief Design Engineer at SK Electronics.

> Hi Felix
> Hope you are doing good. Cindy is our new Head of Sales at SK Electronics. You being our key customer, I wanted Cindy to meet up with you. Are you OK to have a 30 min meeting at 3:00 pm with us on Wednesday, 15th April? We will come over to your office.
> Once you confirm I will schedule it in your calendar.
> Regard
> Tom

On Wednesday, at 3:00 pm, Tom and Robert arrive at Felix's office.

"Hi Felix, good to meet you again" greets Tom. Pointing toward his colleague, Tom continues" This is Cindy. She is the new Head of Sales at SK Electronics."

"Good to meet you Cindy, and congratulations. So, how can I help you guys today?" asks Felix.

"Our CEO kick-started a customer relationship program earlier this year, to develop a deeper understanding of our customer requirements. Cindy and I are here today to find out more about your electronics & circuitry requirements for your current and future line of products."

"We would also appreciate any feedback you may have on our components that have been part of your products for the last 15 years."

"Sure, Tom, and Cindy. I can speak for Yano. Let me start by saying we are pleased with the quality and reliability of the electronic boards we buy from you guys. There are virtually no customer complaints, and it seems our partnership has been mutually beneficial to both our companies. We currently source 2500 circuit boards from you each year, and I do not see that number changing for the next 2 years," says Felix.

"The next year we are coming up with a new line of product, but from a requirements point of view, we will continue to use the same circuit board in the new line. There are no anticipated changes to the design," says Felix.

"My procurement manager tells me that the rates of electronics goods are falling the world over, and despite being a business partner for 15 years you guys have never lowered your rates while we continue to procure components from SK Electronics," Felix adds.

"I can take that up with our CEO and see what best we can work for Yano. I will send you an email in any case.," says Cindy.

The next morning Felix has this email from Cindy in his inbox:

Hi Felix,
Thanks for your hospitality. We had an excellent meeting yesterday and we are glad that SK Electronics have been meeting your requirements. We will strive to continue meeting your expectations.
During our meeting, you did express displeasure on the rates we have been offering on our circuit boards. On my return to the office I had a meeting with our CEO in which **decided we will offer Yano an all-units discount as per schedule below:

Order quantity range	Effective purchase price per unit
0–600 circuit boards	$30.00
600–1500 circuit boards	$29.00
More than 1500 circuit boards	$28.00

I am genuinely hoping you will appreciate the new rates, and am sure Yano and SK Electronics will continue collaborating in the coming years.
Regards
Cindy

Case Study Questions:

(a) If Yano continues procuring circuit boards from SK Electronics, compute the EOQ and the TIC. Use data supplied in the case study for annual demand, carrying rate, and ordering cost.
(b) Compute the feasible and economical order size under the new discount schedule offered by SK Electronics.

Answers:

(a) Optimal order quantity without discount:

Parameter	Value
Optimal order quantity	378
Ordering costs	$1984
Carrying costs	$1984
TIC (not considering purchase costs)	$3968

(b) Feasible and optimal order size under the new discount schedule:

Parameter	Value
Optimal order quantity	600
Ordering costs	\$1250
Carrying costs	\$3045
Purchasing costs	\$72,500
TIC	\$76,795

4.11 Practice Problems

Problem 4.1
Binny Foods manufactures spicy baked potato chips. The baked chips manufacturing process uses three ingredients (items) – potatoes, chilli powder, and salt. Binny's purchases these three items from the same supplier. The inventory attributes for these three items are shown in Table 4.21. The supplier offers a discount of 5% on any individual item that has an order size of 750 kg or more. Compute the EOQ for the three items.
The ordering cost is \$20 per order, and the carrying rate is 15% per year.

Answer:

Item	EOQ at regular price	TIC at regular EOQ	EOQ at 5% discount	TIC at discounted EOQ	Order size decision
Potato	577 kg	20,346	750	19,347	Order size = 750 kg
Chilli powder	73	32,438	750	32,559	Order size = 73 kg
Salt	800	660	800	629	Order Size = 800 kg

Problem 4.2
A manufacturer of plastic bottles needs 1500 kg of raw material each year and sources it from its supplier. Every time the manufacturer places an order, it incurs a cost of \$20. The carrying cost rate is 25% per year. The purchasing price of raw material is based on the size of the order as shown below:

Cost per kg : \$1.00 for order size between 0 and 499 kg.
: \$0.99 for order size between 500 and 999 kg.
: \$0.98 for order size above 1000 kg.

Assuming all-units discount, determine the feasible and optimal order quantity.

Table 4.21 Data for Problem 4.1

Item name	Annual demand	Unit price
Potatoes	5000 kg	\$4 per kg
Chilli powder	800 kg	\$40 per kg
Salt	1200 kg	\$0.5 per kg

Answer:

EOQ strategy	EOQ	Order cost	Carrying cost	Inventory investment	Total inventory cost
$1.00	490	61	61	1500	$1622
$0.99	500	61	61	1485	$1607
$0.98	1000	30	123	1470	$1623

EOQ of 500 is optimal, with a TIC of $1607.

Problem 4.3
An electronics toy manufacturer requires 1200 small power DC motors each year and sources it from its supplier. Every time the manufacturer places an order for procuring the motors, it incurs a cost of $20. The carrying cost rate is 15%. The purchasing price of circuit board is based on the size of the order as shown below:

Cost per motor : $5.00 for order size between 0 and 300 nos.
: $4.50 for order size between 300 and 600 nos.
: $4.00 for order size above 600 nos.

Use incremental discount method to determine feasible and optimal order size.

Answer

EOQ strategy	EOQ	Feasible?	Total inventory cost
$5.0	252	Yes	$6190
$4.5	777	No	$5936
$4.0	1371	Yes	$5656

The optimal and feasible quantity is 1371 units.

Problem 4.4
Every year a retailer buys 500 units of a submersible pump (Model No. SX-25) directly from its manufacturer at $240 per unit. The retailer incurs an ordering cost of $100 per order and a carrying rate of 35% per year. The manufacturer of the submersible pump has just announced a $10 price increase effective in 2 weeks' time. Compute the size of order if the retailer would like to utilize this one-time opportunity to reduce future cost of sales.

Answer
Current EOQ (before price increase)

$$\sqrt{\frac{2 \times 500 \times 35}{0.35 \times 240}} = 34 \text{ units}$$

Quantity computed using discounted price:

$$Q_d = \frac{10 \times 500}{240 \times 0.35} + \frac{34 \times 250}{240} = 60 + 35 = 95 \text{ units}$$

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Chapter 5
Lot-Sizing Heuristics

5.1 Introduction

Corn flour additive, a customized *masala*,[1] is used by Rosetta's to make corn-flavored tortillas. Rosetta's buys this each month from their preferred vendor. The demand for this item varies each month. It peaks during primavera and starts dipping as summer approaches. It increases once again during the rainy season. However, the demand for this item is known at least 6 months in advance, thanks to Rosetta's demand estimation system for this product that forecasts fairly accurately the monthly demand. The demand (in kilograms) over the next 6 months (January through June) is as shown below.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (kg)	36	60	85	11	39	75

Considering the fact that demand varies each month, what quantity should Rosettas order, and when? These are the questions that will be addressed in this chapter.

In the running example (presented in the box), the demand for the corn flour additive varies during each period. It invalidates one of the key assumptions of applying the classical *EOQ* formula for computing economic lot size – that of

[1]A mixture of spices.

D. Shenoy, R. Rosas, *Problems & Solutions in Inventory Management*,
DOI 10.1007/978-3-319-65696-0_5

constant demand. The *EOQ* formula, therefore, cannot be used. In such a situation we can apply one of the several lot-sizing heuristics that have been developed. Heuristics provide a practical method of arriving at a solution that is feasible but not necessarily optimal. The following terminologies have been used in this chapter:

- Order Horizon: This is the number of periods for which an order is expected to meet the demand. If we place an order at the beginning of a period, and the size of the order is big enough to meet the demand for, say, three periods, then we say the order horizon is three periods.
- Planning Horizon: This is the finite number of periods over which a lot-sizing problem is to be solved. We know the demand for corn flour additive for the next 6 months (January–June), and we need to determine the order size. The planning horizon, in the case of the running example, is 6 periods (months).
- Closeness Factor: An order cost is incurred each time an order is placed. A holding cost, based on the number of units of items held in the inventory, is also incurred. The closeness factor is the parameter computed as the difference of the order cost to the total holding costs for a given iteration. While comparing closeness factors between iterations, the one that is smaller corresponds to the optimal order.
- Iteration: This is a repetition of a step with different parameters. In the context of this chapter, the term iteration refers to computing the inventory costs for different periods.

5.1.1 Assumptions

The following assumptions have been made while discussing lot-sizing heuristics in this chapter:

- The demand is for a single item.
- The demand for the item under consideration is known.
- Replenishment lead time is zero.
- Orders are placed at the beginning of a period, i.e., the first day of the period. Items ordered are received in full instantaneously. There are no defective items received.
- The size of the order placed is such that it completely meets the demand for that period or an integral number of periods.
- There are no capacity constraints, i.e., storage capacity is infinite.

5.2 Lot-Sizing Heuristics

While several heuristics can be found in literature (see Sreekumar et al. 1991; Silver et al. 1998; Nahmias 2005), we will restrict our discussion to the application of the following that are more popular in the industry:

Table 5.1 Demand for Corn flour additive

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (kg)	36	60	85	11	39	75

(a) Lot-for-Lot
(b) Part-Period Balancing
(c) Silver-Meal
(d) Least Unit Cost, and
(e) Wagner-Whitin

Each of the above methods is discussed in detail in the following sections.

5.2.1 Lot-for-Lot Heuristic

Lot-for-lot is one of the simplest of heuristics to implement. As the name suggests, this heuristic is based on the philosophy that we place an order for a quantity that is equal to the demand for any given period (month, in this case).

Consider the demand for corn flour additive as shown in Table 5.1. If we are to use lot-for-lot method, then we would place an order for 36 units at the beginning of January, 60 units at the beginning of February, 85 units at the beginning of March, and so on. Let us now compute the total inventory cost using the Lot-for-Lot method. We are given the following information:

- Order cost C_o is \$80 per order
- Holding cost C_h is \$1.75 per unit per period

Iteration 1[2]

We start the solution process by placing an order for 36 units, in January.

Place an order at the beginning of January for 36 units that will completely meet the demand for that month

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

[2] See terminology in Sect. 5.1.

If we place an order for 36 units at the beginning of January, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	We assume that the ordered quantity will be received in full at the beginning of January. Thus, the inventory level at the beginning of January would be 36 units and that at the end of January would be 0. The average inventory level in January would, therefore, be $= \frac{36 + 0}{2}$ $= 18$ units The holding cost for January would thus be 18 × $1.75 or $31.5

Iteration 2

Next, we place an order for 60 units at the beginning of February that will completely meet the demand for that month.

Place an order at the beginning of February for 60 units that will completely meet the demand for that month

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 60 units at the beginning of February, we incur the following costs:

Type of Cost	Description
Ordering Cost	Since an order is placed, we incur an ordering cost of $80
Holding Cost	We assume that the ordered quantity will be received in full at the beginning of February. Thus, the inventory level at the beginning of February would be 60 units and that at the end of February would be 0. The average inventory level in February would, therefore, be $= \frac{60 + 0}{2}$ $= 30$ units The holding cost for February would thus be 30 × $1.75 or $52.5

Using the same technique, we find the total inventory costs assuming that an order is placed at the beginning of each month that would completely satisfy the demand for that month. The total inventory costs would be as shown in Table 5.2.

So, if we are to use the lot-for-lot method, the total inventory cost would be $747.75.

Table 5.2 Lot-for-lot solution summary

Month	Ordering cost ($)	Holding cost ($)	Total cost ($)
January	$80	$31.50	$111.50
February	$80	$52.50	$132.50
March	$80	$74.38	$154.38
April	$80	$9.63	$89.63
May	$80	$34.13	$114.13
June	$80	$65.63	$145.63
Total	$480	$267.751	$747.75

Solved Problem 5.1
Monthly demand for an item over 6 months is 32, 19, 12, 15, 23, 12 units, respectively. Using lot-for-lot method, determine the total inventory cost if the holding cost is $1.5 per unit per month and the ordering cost is $40 per order.

Solution

Alternative	Order cost ($)	Holding cost ($)	Total cost ($)
Order 32 to cover the demand for Month 1	40	24.0	64.0
Order 19 to cover the demand for Month 2	40	14.3	54.3
Order 12 to cover the demand for Month 3	40	9.0	49.0
Order 15 to cover the demand for Month 4	40	11.3	51.3
Order 23 to cover the demand for Month 5	40	17.3	57.3
Order 12 to cover the demand for Month 6	40	9.0	49.0

If we use the lot-for-lot method, the total inventory cost would be $324.8.

A note on calculation of holding cost
When calculating the holding cost, we assume that the ordered quantity will be received in full at the beginning of a given period. Thus, the inventory level at the beginning of a period would be D_1units and that at the end of the period would be 0. The average inventory is $\frac{D_1+0}{2}$, and the holding cost is $\frac{D_1+0}{2}C_h$. (Srinivasan 2010). This method of calculation is consistent with the approach taken in other chapters in this book.

5.2.2 *Part-Period Balancing*

Part-Period Balancing (PPB) is a heuristic that is based on the concept of balancing the order cost with the holding cost. This method requires calculating the holding costs as a function of the number of periods the current order spans, or the order horizon (Nahmias 2005). An iteration is completed when the holding costs for an order exceed the order cost. The order quantity that has a total holding cost closest to the order cost is considered to be optimal. This can be described mathematically as follows:

Let

- $d_1, d_2, d_3, \cdots, d_n$ be the demand for an item in a period, spanning a horizon of n periods
- C_o be the ordering cost
- C_h be the holding cost per item per period

Consider an order horizon of j periods. The total holding cost over this order horizon is given by

$$C_\mathrm{h} \sum_{i=1}^{j} \frac{(2i-1)d_i}{2} \tag{5.1}$$

where $i \in j$. The closeness factor, C_r, for the order horizon of j periods is given by

$$C_\mathrm{r} = \left| C_\mathrm{o} - C_\mathrm{h} \sum_{i=1}^{j} \frac{(2i-1)d_i}{2} \right| \tag{5.2}$$

If the C_r over an order horizon of $(j-1)$ periods is smaller (i.e., closer to C_o) than that of j periods, then we set the optimal order horizon to $(j-1)$ periods. The optimal order quantity in this case would be

$$Q_{j-1} = \sum_{i=1}^{j-1} d_i \tag{5.3}$$

In the following section, we will learn the PPB heuristic by applying it to the problem presented in the running example. We will use the same cost parameters that we used earlier, i.e.,

- Ordering cost $C_\mathrm{o}$is \$80 per order
- Holding cost $C_\mathrm{h}$is \$1.75 per unit per period

Iteration 1.1

We start the solution process with an order size of 36 units that will satisfy the demand for January.

Place an order in January that completely meets the demand for that month

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 36 units at the beginning of January, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	We assume that the ordered quantity will be received in full at the beginning of January. Thus, the inventory level at the beginning of January would be 36 units and that at the end of January would be 0. The average inventory level in January would, therefore, be $= \frac{36+0}{2}$ =18 units The holding cost for January would thus be 18 × \$1.75 or \$31.50

In this case, the holding cost (\$31.5) is less than the order cost (\$80). Therefore, the next step would be to set the order horizon to 2 months – January and February – and place one order, at the beginning of January, that would meet the requirements for both January and February. We perform the next iteration using this information.

Iteration 1.2

Order horizon is 2 months. We combine the demands for January and February. Place one order at the beginning of January

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for January is 36 units and that for February is 60 units. If we place an order for 96 units at the beginning of January, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of January, the inventory level at the beginning of January is 96 units. The inventory level at the end of January is 60 units. So the average inventory level in January would be $= \frac{96+60}{2}$ =78 units The inventory level at the beginning of February would be 60 units and at the end of February would be 0. So, the average inventory level in February would be $= \frac{60+0}{2}$ =30 units The total holding cost for the months of January and February would, therefore, be: =(78 + 30) × \$1.75 = \$189.00

Table 5.3 Closeness factor for January–Febuary order horizon

Parameter	January	January + February
C_r	48.5	109
Q	36	96

At this point we notice that the holding cost ($189) has exceeded the cost of placing one order ($80). We now need to check the C_r values. The calculation of C_r can be accomplished as follows:

From Iteration 1.1, we see that the holding cost for ordering 36 units in January is $31.5. The C_r for this option is |$80 – $31.5| = $48.5.

From Iteration 1.2, we see that the holding cost for ordering 96 units in January is $189. The C_r for this option is |$80 – $189| = $109. It should be noted that we are interested only in the difference between the holding costs and the ordering cost. Therefore, we ignore the sign.

As seen from Table 5.3, C_r for the lot size of 36 units is smaller for the two order horizons – January and January + February (or the total holding cost is closer to the ordering cost). We can, therefore, conclude that it is economical to place an order for 36 units in January, which will help meet the demand for January alone.

Iteration 2.1

Since the demand for February was not included in the previous lot size, we start a new iteration after setting the starting period to February. The demand for February is 60 units.

Place one order at the beginning of February for 60 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 60 units at the beginning of February, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of February, the inventory level at the beginning of February would be 60 units. The inventory level at the end of February would be 0. The average inventory level in February would, therefore, be $= \frac{60+0}{2}$ =30 units. Thus, the holding cost for February would be 30 × $1.75 or $52.50.

The holding cost ($52.5) is less than the order cost ($80). Therefore, the next step is to set an order horizon to 2 months – February and March – and place one order, at the beginning of February, that would meet the requirement for both these months.

Iteration 2.2

In this iteration, we set the order horizon to 2 months – February and March, and place an order for 145 units at the beginning of February.

Combine the demands for February and March.
Place one order at the beginning of February

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 145 units at the beginning of February, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 145 units in full at the beginning of February, the inventory level at the beginning of February is 145 units. The inventory level at the end of February is 85 units. So the average inventory level in February would be $= \frac{145 + 85}{2}$ =115 units The inventory level at the beginning of March would be 85 units and at the end of March would be 0. So the average inventory level in March would, therefore, be $= \frac{85 + 0}{2}$ =42.50 units The total holding cost for the months of February and March would, therefore, be $=(115 + 42.50) \times \$1.75 = \$275.63$

At this stage, we notice that the total holding cost (*$*275.625) has exceeded the order cost ($80). We now need to check the C_r values.

As seen from Table 5.4, the C_r for February is closer than that of the combined February and March order horizon. We can, therefore, conclude that it is economical to place an order for 60 units in February, which will help meet the demand for February only.

Table 5.4 Closeness factor for February–March order horizon

Parameter	February	February + March
C_r	27.5	195.625
Q	60	145

Iteration 3.1

The demand for March was not included as part of the order placed in February. We, therefore, start a new iteration after setting the initial period to March. The demand for March is 85 units.

Place an order at the beginning of March that will completely meet the demand for that month

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 85 units at the beginning of March, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80
Holding cost	Since we receive the ordered quantity in full at the beginning of March, the inventory level at the beginning of March would be 85 units. The inventory level at the end of March would be 0. The average inventory level in March would, therefore, be $= \frac{85+0}{2}$ =42.5 units Thus, the holding cost for March would be 42.5 × \$1.75 or \$74.38

The holding cost (\$74.375) is less than the order cost (\$80). Therefore, the next step is to set an order horizon to 2 months – March and April – and place one order, at the beginning of March, which would meet the requirements for both these months.

Iteration 3.2

Combine the demands for March and April. Place one order at the beginning of March

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for March is 85 units and April is 11 units. If we place an order for 96 units at the beginning of March, the following costs would be incurred:

Table 5.5 Closeness factor for March–April order horizon

Parameter	March	March + April
C_r	5.62	23.25
Q	85	96

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of March, the inventory level at the beginning of March is 96 units. The inventory level at the end of March is 11 units. So the average inventory level in March would be $= \frac{96 + 11}{2}$ =53.50 units The inventory level at beginning of April would be 11 units and at the end of April would be 0. So the average inventory level in April would, therefore, be $= \frac{11 + 0}{2}$ =5.5 units The total holding cost for months of March and April would, therefore, be $=(53.50 + 5.50) \times \$1.75 = \$103.25$

At this stage we notice that the total holding cost (\$103) has exceeded the order cost (\$80). We now need to check the C_r values.

As seen from Table 5.5, since C_r for the lot size of 85 units is smaller (meaning, the holding cost is closer to the ordering cost), we can conclude that it is economical to place an order for 85 units in March, which will help meet the demand for March only.

Iteration 4.1

The demand for April was not included as part of the order placed in March. We, therefore, start a new iteration after setting the initial period to April. The demand for April is 11 units.

Place one order at the beginning of April that would meet the demand for April only

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 11 units at the beginning of April, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of April, the inventory level at the beginning of April would be 11 units. The inventory level at the end of April would be 0. The average inventory level in April would, therefore, be $= \frac{11 + 0}{2}$ =5.5 units Thus, the holding cost for April would be 5.5 × $1.75 or $9.63.

The holding cost ($9.625) is less than the order cost ($80). Therefore, the next step is to set an order horizon to 2 months – April and May – and place one order, at the beginning of April, that would meet the requirements for both these months.

Iteration 4.2

Combine the demands for April and May. Place one order at the beginning of April

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for April is 11 units and that for May is 39 units. If we place an order for 50 units at the beginning of April, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 50 units in full at the beginning of April, the inventory level at the beginning of April is 50 units. The inventory level at the end of April is 39 units. So the average inventory level in April would be $= \frac{50+39}{2}$ =44.50 units The inventory level at beginning of May would be 39 units and at the end of May would be 0. So the average inventory level in May would, therefore, be $= \frac{39 + 0}{2}$ =19.5 units The total holding cost for months of April and May would, therefore, be =(44.50 + 19.50) × $1.75 = $112.00

At this stage, we notice that the total holding cost ($112) has exceeded the order cost ($80). We now need to compare the C_r values.

As seen from Table 5.6, since C_r for the lot size of 50 units is smaller (meaning, the holding cost is closer to the ordering cost), we can conclude that it is economical

Table 5.6 Closeness Factor for April–May order horizon

Parameter	April	April + May
C_r	70.37	32
Q	11	50

to place an order for 50 units in April, which will help meet the demand for April and May.

Iteration 5.1

We next start a new iteration after setting the starting period to June.

Place one order at the beginning of June

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for June is 75 units. If we place an order for 75 units at the beginning of June, the inventory cost incurred would be as follows:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 75 units in full at the beginning of June, the inventory level at the beginning of June is 75 units. The inventory level at the end of June is 0 units. So the average inventory level in June would be $= \frac{75 + 0}{2}$ $= 37.50$ units The total holding cost for month of June would, therefore, be $= (37.5) \times \$1.75 = \$65.625$

Since we currently do not know the demands for the month beyond June we can stop the solution procedure. We assume that it is economical to place an order for 75 units at the beginning of June. The solution to the lot-sizing problem presented in the running example, based on part-period balancing method, is as shown in Table 5.7

The solution produced by the part-period balancing method suggests that we place an order for 36 units in January, 60 units in February, 85 units in March, 50 units in April (that would cover the demand for April and May), and 75 units in June. The total inventory cost for the planning horizon is \$736.

Solved Problem 5.2

Monthly demand for an item over 6 months is 32, 19, 12, 15, 23, and 12, units respectively. Using the Part Period Balancing method, determine the total inventory cost if the holding cost is \$1.5 per unit per month and the ordering cost is \$40 per order.

Table 5.7 Part-period balancing solution summary

Month	Order quantity	Ordering cost	Holding cost	Inventory cost
January	36	$80	$31.50	$111.50
February	60	$80	$52.50	$132.50
March	85	$80	$74.38	$154.38
April	50	$80	$112.00	$192.00
May				
June	75	$80	$65.63	$145.63
Total Inventory Cost (TIC)		$400	$336.01	$736.01

Solution

Alternative	Order cost ($)	Holding cost ($)	Closeness ratio	Remarks
Order 32 units to cover the demand for Month 1	40	24.0		Continue
Order 51 units to cover the demand for Month 1 and Month 2	40	66.75	Min for 32 units	Stop since holding cost has exceeded ordering cost
Based on closeness ratio, it is optimal to order 32 units in Month 1				
Order 19 units to cover the demand for Month 2	40	14.25		Continue
Order 31 units to cover the demand for Month 2 and Month 3	40	41.25	Min for 31 units	Stop since holding cost has exceeded ordering cost
Based on closeness ratio, it is optimal to order 31 units in Month 2 (to cover demands for Mmonth 2 and Month 3).				
Order 15 units to cover the demand for Month 4	40	11.25		Continue
Order 38 units to cover the demand for Month 4 and Month 5	40	63	Min for 38 units	Stop since holding cost has exceeded ordering cost
Based on closeness ratio, it is optimal to order 38 units in Month 4 (to cover demands for Month 4 and Month 5).				
Order 12 units to cover the demand for Month 4	40	9.0		Stop since information about future demand is not available
Place an order for 12 units in Month 6				
Total Inventory Cost: 40 + 24 + 40 + 41.25 + 40 + 63 + 40 + 9 = $297.25				

5.2.3 Silver-Meal Heuristic

The Silver-Meal heuristic (Silver and Meal 1973) involves computing the *total inventory costs per period*. The key idea behind this heuristic is that the total relevant costs per unit time for the duration of the replenishment quantity are minimized (Silver et al. 1998).

Let

- $d_1, d_2, d_3, \ldots, d_n$ be the demand for an item in the *j*th period over a planning horizon of n periods
- C_o be the ordering cost per order
- C_h be the holding cost per item per period

Consider an order horizon of *j* periods. The total inventory cost over this order horizon is given by

$$C_o + C_h \sum_{i=1}^{j} \frac{(2i-1)d_i}{2} \tag{5.4}$$

where $i \in j$

The per period cost (PPC) for the order horizon of *j* periods is given by

$$\frac{C_o + C_h \sum_{i=1}^{j} \frac{(2i-1)d_i}{2}}{\sum_{i=1}^{j} i} \tag{5.5}$$

If the total inventory cost over an order horizon of *j* periods is greater than that of $(j-1)$ periods, then we set the optimal order horizon to $(j-1)$ periods. The optimal order quantity in this case would be

$$Q_{j-1} = \sum_{i=1}^{j-1} d_i \tag{5.6}$$

We continue the iterations until the end of the planning horizon.

Let us now solve the inventory problem for the Rosetta's running example using the Silver-Meal heuristic. We will use the same cost parameters that we used earlier, i.e.,

- Ordering cost C_o is \$80 per order
- Holding cost C_h is \$1.75 per unit per period

Iteration 1.1

We start the solution process by creating an order for 36 units that would entirely meet the demand for January (and January only).

Place one order at the beginning of January
that would meet the demand in its entirety

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 36 units at the beginning of January, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 36 units. The inventory level at the end of January would be 0. The average inventory level in January would, therefore, be $= \frac{36+0}{2}$ $=18$ units Thus, the holding cost for January would be $18 \times \$1.75$ or \$31.5.
Total inventory cost	The total inventory cost is $\$80 + \$\ 31.5$.
Per Period Cost (PPC)	Since this cost is incurred over one period (January), the PPC is $= \frac{\$111.5}{1}$ $=\$111.50$

Iteration 1.2

The next step would be to set an order horizon to 2 months – January and February – and place one order, at the beginning of January, that would meet the requirements for both these months.

Combine the demands for January and February and place one order at the beginning of January.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for January is 36 units and that for February is 60 units. If we place an order for 96 units at the beginning of January, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of January, the inventory level at the beginning of January is 96 units. The inventory level at the end of January is 60 units. So, the average inventory level in January would be $= \frac{96+60}{2}$ $=78$ units The inventory level at the beginning of February would be 60 units and at the end of February would be 0. So, the average inventory level in February would, therefore, be $= \frac{60+0}{2}$

(continued)

Type of cost	Description
	=30 units The total holding cost for the months of January and February would, therefore, be $=(78+30)\times \$1.75=\$189$
Total Inventory Cost	The total inventory cost is $\$80+\$189=\$269$
Per Period Cost (PPC)	Since this cost is incurred over two periods (January and February), the PPC is $=\frac{\$269}{2}$ $=\$134.5$

Fig 5.1 shows the comparison of the PPC values. Since the PPC for the order horizon of January and February taken together is (134.5) greater than the PPC for the order horizon for January alone (111.5), the Silver-Meal heuristic suggests that it is economical to place an order for 36 units at the beginning of January.

Iteration 2.1

In this iteration we set the order horizon to February since the demand for this month was not included in the previous order.

Place an order for 60 units at the beginning of February.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

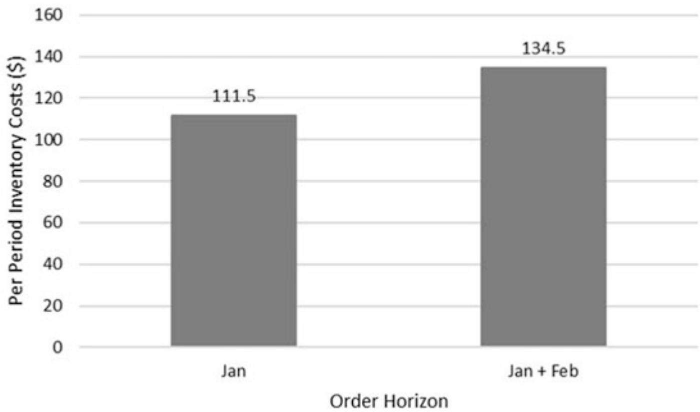

Fig. 5.1 PPC for January–February order horizon

The demand for February is 60 units. If we place an order for 60 units at the beginning of February, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 60 units in full at the beginning of February, the inventory level at the beginning of February is 60 units. The inventory level at the end of February is 0 units. So, the average inventory level in February would be $= \frac{60 + 0}{2}$ $= 30$ units The total holding cost for the months of February would, therefore, be $= (30) \times \$1.75 = \$52.50$
Total inventory cost	The total inventory cost is $\$80 + \$52.50 = \$132.50$
Per Period Cost (PPC)	Since this cost is incurred over one period (February), the PPC is $= \frac{\$132.50}{1}$ $= \$132.50$

The next step is to combine the demands for February and March, and compare the PPC for this horizon with that of February.

Iteration 2.2

In this iteration, we set the order horizon to February and March

Please an order for 145 units at the beginning of February.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for February is 60 units and for March is 85 units. If we place an order for 145 units at the beginning of February, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 145 units in full at the beginning of February, the inventory level at the beginning of February would be 145 units. The inventory level at the end of February would be 85 units. So, the average inventory level in February would be $= \frac{145 + 85}{2}$ $= 115$ units

(continued)

Type of cost	Description
	The inventory level at the beginning of March would be 85 units, and at the end of March it would be 0 units. So, the average inventory level in March would be $= \dfrac{85 + 0}{2}$ $= 42.5$ units The total holding cost for February and March would be $= (115 + 42.5) \times \$1.75 = \$275.625$
Total inventory cost	The total inventory cost is $\$80 + \$275.63 = \$355.625$
Per Period Cost (PPC)	Since this cost is incurred over two periods (February and March), the PPC is $= \dfrac{\$355.625}{2}$ $= \$177.81$

Figure 5.2 shows the comparison of the PPC values. Since the PPC for the order horizon of February and March taken together ($177.81) is greater than the PPC for the order horizon for February alone ($132.5), the Silver-Meal heuristic suggests that it is economical to place an order for 60 units at the beginning of February.

Iteration 3.1

We start the next iteration by setting the start period to March. The total demand for March is 85 units. If we place an order for 85 units at the beginning of March, the following costs would be incurred:

Place one order to meet the demand for March at the beginning of March for 85 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

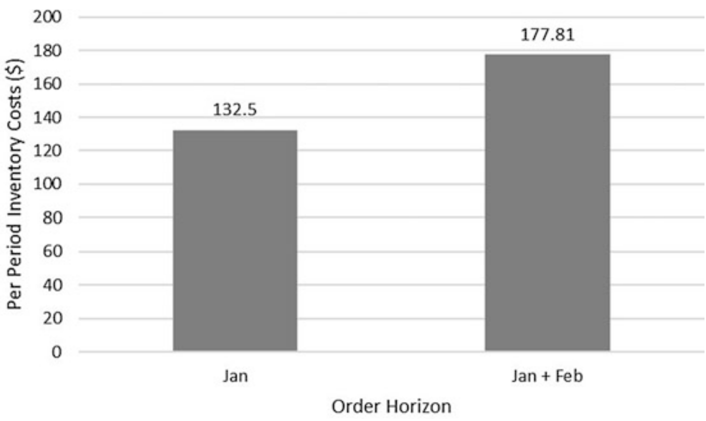

Fig. 5.2 PPC for February–March order horizon

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 85 units in full at the beginning of March, the inventory level at the beginning of March would be 85 units and at the end of March would be 0. So, the average inventory level in March would, therefore, be $=\frac{85+0}{2}$ =42.5 units The total holding cost for March would, therefore, be $=(42.5) \times \$1.75 = \$74.375$
Total inventory cost	The total inventory cost is $\$80 + \$74.375 = \$154.375$
Per Period Cost (PPC)	Since this cost is incurred in March, the PPC is $=\frac{\$154.375}{1}$ =\$154.38

For the next iteration, we batch the demands for March and April and set the start period to March.

Iteration 3.2

The next step is to take an order horizon of 2 months – March and April.

Combine the demands for March and April. Place one order at the beginning of March for 96 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for March is 85 units and that for April is 11 units. If we place an order for 96 units at the beginning of March, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of March, the inventory level at the beginning of March is 96 units. The inventory level at the end of March is 11 units. So, the average inventory level in March would be $=\frac{96+11}{2}$ =53.5 units The inventory level at the beginning of April would be 11 units and that at the end of April would be 0. So, the average inventory level in April would, therefore, be $=\frac{11+0}{2}$ =5.5 units

(continued)

Type of cost	Description
	The total holding cost for the months of March–April would, therefore, be $=(53.5+5.5)\times \$1.75=\$103.25$
Total inventory cost	The total inventory cost is $\$80+\$103.25=\$183.25$
Per Period Cost (PPC)	Since this cost is incurred over two periods (March–April), the PPC is $=\frac{\$183.25}{2}$ $=\$91.63$

At this stage, we see that the PPC for order horizon March–April is lower than the PPC for March. Therefore, we can continue batching demands. We next set the order horizon to March–April–May with the order being received at the beginning of March.

Iteration 3.3
We start the next iteration considering demand for March–April–May.

Place one order at the beginning of March for 135 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for March is 85 units, that for April is 11 units, and that for May is 39 units. If we place an order for 135 units at the beginning of March, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 135 units in full at the beginning of March, the inventory level at the beginning of March would be 135 units. The inventory level at the end of March is 50 units. So, the average inventory level in March would be $=\frac{135+50}{2}$ $=92.5$ units The inventory level at the beginning of April would be 50 units. The inventory level at the end of April is 39 units. So, the average inventory level in April would be $=\frac{50+39}{2}$ $=44.5$ units The inventory level at the beginning of May would be 39 units. The inventory level at the end of May is 0 units.

(continued)

Type of cost	Description
	So, the average inventory level in May would be $=\dfrac{39+0}{2}$ =19.5 units The total holding cost for this order horizon March–April–May would be =(92.5 + 44.5 + 19.5) × \$1.75 = \$273.88
Total inventory cost	The total inventory cost is \$80 + \$273.875 = \$353.875
Per Period Cost (PPC)	Since this cost is incurred over three periods (March–April–May), the PPC is $=\dfrac{\$353.875}{3}$ = \$117.96

At this stage, we see that the PPC for order horizon March–April–May is greater than the PPC for March–April (see Fig. 5.3). Based on the Silver-Meal heuristic we can conclude that it is optimal to place one order of 96 units at the beginning of March. We next set the order horizon to May with the order being received at the beginning of May

Iteration 4.1

The demand for May is 39 units. If we place an order for 39 units at the beginning of May, the following costs would be incurred:

Place one order at the beginning of May for 39 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

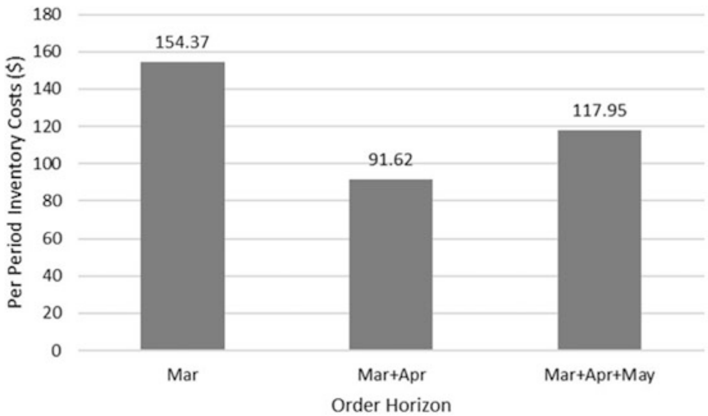

Fig. 5.3 PPC for March–April–May order horizon

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	The inventory level at the beginning of May would be 39 units. The inventory level at the end of May is 0 units. So, the average inventory level in May would be $=\frac{39+0}{2}$ $=19.5$ units The total holding cost for this order horizon of May would be $=(19.5)\times\$1.75=\$34.13$
Total inventory cost	The total inventory cost is $\$80+\$34.125=\$114.125$
Per period cost (PPC)	Since this cost is incurred over 1 period (May), the PPC is $=\frac{\$114.125}{1}$ $=\$114.13$

Next we set the order horizon to two periods – May and June. The total demand for May and June is 114 units.

Iteration 4.2

Combine the demands for May, and June. Place one order at the beginning of May for 114 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 114 units at the beginning of May, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 114 units in full at the beginning of May, the inventory level at the beginning of May would be 114 units. The inventory level at the end of May would be 75 units. So, the average inventory level in May would be $=\frac{114+75}{2}$ $=94.5$ units The inventory level at the beginning of June would be 75 units and at the end of June would be 0. So, the average inventory level in June would, therefore, be $=\frac{75+0}{2}$ $=37.5$ units

(continued)

Type of cost	Description
	The total holding cost for the months of May and June would, therefore, be $=(94.5+37.5)\times\$1.75=\$231$
Total inventory cost	The total inventory cost is $\$80+\$231=\$311$
Per period cost (PPC)	Since this cost is incurred over two periods (May and June), the PPC is $=\dfrac{\$311}{2}$ $=\$155.50$

Now, it is noticed (from Fig. 5.4) that the PPC for order horizon of May–June is greater than the PPC for May. Therefore, the optimal order quantity is the demand for May alone (39 units).

Iteration 5.1

We next start a new iteration after setting the starting period to June. The demand for June is 75 units. Since we currently do not know the demands for the months beyond June we can stop the lot-sizing procedure. We assume that it is economical to place an order for 75 units at the beginning of June.

Place one order at the beginning of June for 75units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 75 units at the beginning of June, the following costs would be incurred:

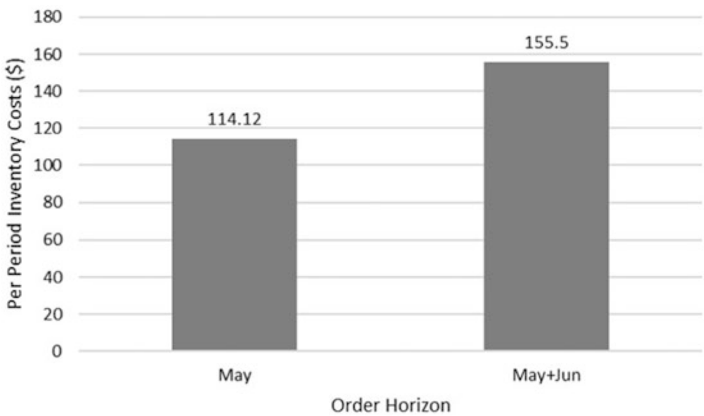

Fig. 5.4 PPC for May–June order horizon

Table 5.8 Silver-Meal solution summary

Month	Order quantity	Ordering cost	Holding cost	Inventory cost
January	36	$80	$31.50	$111.50
February	60	$80	$52.50	$132.50
March	96	$80	$103.25	$183.25
April				
May	39	$80	$34.13	$114.13
June	75	$80	$65.62	$145.62
Total inventory cost (TIC)		$400	$287.00	$687.00

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 75 units in full at the beginning of June, the inventory level at the beginning of June would be 75 units. The inventory level at the end of June would 0 units. So, the average inventory level in June would be $=\frac{75+0}{2}$ =37.5 units The total holding cost for the month of June would, therefore, be $=(37.5)\times\$1.75=\$65.63$

Since we do not have the demands for the subsequent months, we stop the solution process here. The solution summary to this lot-sizing problem based on the Silver-Meal method is shown in Table 5.8.

The solution produced by the Silver-Meal method suggests that we place an order for 36 units in January, 60 units in February, 96 units in March (that would cover demand for March and April), 39 units in May, and 75 units in June. The total inventory cost for the planning horizon is $687.

Solved Problem 5.3

Monthly demand for an item over 6 months is 32, 19, 12, 15, 23, and 12, units respectively. Using Silver-Meal method, determine the total inventory cost if the holding cost is $1.5 per unit per month and the ordering cost is $40 per order.

Solution

Alternative	Order cost ($)	Holding cost ($)	Per Period cost	Remarks
Order 32 units to cover the demand for Month 1	40	24.0	64.0	Continue
Order 51 units to cover the demand for Month 1 and Month 2	40	66.75	53.38	Since PPC is lower than the previous iteration, continue batching demand
Order 63 units to cover the demand for Month 1, Month 2, and Month 3	40	111.75	50.59	Since PPC is lower than the previous iteration, continue batching demand

(continued)

Alternative	Order cost ($)	Holding cost ($)	Per Period cost	Remarks
Order 78 units to cover the demand for Month 1 through Month 4	40	190.5	57.63	STOP
Since PPC for Month 1 through 3 is the least, the optimal lot size is 63 units for the order to be placed at the beginning of Month 1.				
Order 15 units to cover the demand for Month 4	40	11.25	51.25	Continue
Order 38 units to cover the demand for Month 4 and Month 5	40	63.0	51.5	STOP
Since PPC for Month 4 is the least, the optimal lot size is 15 units for the order to be placed at the beginning of Month 4.				
Order 23 units to cover the demand for Month 5	40	17.25	57.25	Continue
Order 37 units to cover the demand for Month 4 and Month 5	40	44.25	41.13	Continue
Since we do not have demand information for the future periods, we assume it is optimal to order 35 units at the beginning of Month 5.				
Total Inventory Cost: 40 + 111.75 + 40 + 11.25 + 40 + 44.25 = $287.25				

5.2.4 *Least Unit Cost Heuristic*

In this section, we use the least unit cost method to determine the order quantity and total inventory costs. This method is very similar to the Silver-Meal heuristic, except that while Silver-Meal heuristic uses the number of periods to find the per period cost, the least unit cost method uses the number of units in an order horizon to determine the least unit cost (Nahmias 2005).

Let

- $d_1, d_2, d_3, \cdots, d_n$ be the demand for an item in the jth period over a planning horizon of n periods
- C_o be the ordering cost
- C_h be the holding cost per ítem per period

Consider an order horizon of j periods. The total inventory cost over this order horizon is given by

$$C_o + C_h \sum_{i=1}^{j} \frac{(2i-1)D_i}{2} \tag{5.7}$$

where $i \in j$

The least unit cost for an order horizon of j periods is given by

$$\frac{C_o + C_h \sum_{i=1}^{j} \frac{(2i-1)d_i}{2}}{\sum_{i=1}^{j} d_i} \tag{5.8}$$

If the total inventory cost in an order horizon of j periods is greater than that of $(j-1)$ periods, then we set the optimal order horizon to $(j-1)$ periods. The optimal order quantity in this case would be

$$Q_{j-1} = \sum_{i=1}^{j-1} D_i \tag{5.9}$$

We continue the iterations until the end of the planning horizon. Let us now solve the Rosetta's inventory problem using the least unit cost method.

Iteration 1.1

We start the iteration by creating an order for 36 units that would entirely meet the demand for January, and January only.

Place an order at the beginning of January for 36 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 36 units at the beginning of January, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 36 units. The inventory level at the end of January would be 0. The average inventory level in January would, therefore, be $= \frac{36+0}{2}$ $=18$ units Thus, the holding cost for January would be $18 \times \$1.75$ or \$31.5.
Total inventory cost	The total inventory cost is \$80 + \$ 31.5
Per Unit Cost (PUC)	The total demand for this period is 36 units. The Per Unit Cost (PUC) is $= \frac{\$111.5}{36}$ $= \$3.097$

Iteration 1.2

The next step would be to set an order horizon to 2 months – January and February – and place one order at the beginning of January that would meet the requirements for both these months.

Combine the demands for January and February and place one order at the beginning of January.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for January is 36 units and that for February is 60 units. If we place an order for 96 units at the beginning of January, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of January, the inventory level at the beginning of January is 96 units. The inventory level at the end of January is 60 units. So, the average inventory level in January would be $= \frac{96 + 60}{2}$ $= 78$ units The inventory level at the beginning of February would be 60 units and at the end of February would be 0. So, the average inventory level in February would, therefore, be $= \frac{60 + 0}{2}$ $= 30$ units The total holding cost for the months of January and February would, therefore, be $= (78 + 30) \times \$1.75 = \$189$
Total inventory cost	The total inventory cost is \$80 + \$189 = \$269
Per unit cost (PUC)	The total demand for January and February is 96 units the PUC is therefore $= \frac{\$269}{96}$ $= \$2.802$

Since the per unit cost (PUC) for the order horizon of January and February taken together is less than the PUC for the order horizon for January alone, we continue this iteration by combining the demand for January, February, and March in one order, placed at the beginning of January.

Iteration 1.3

Combine the demands for January, February and March. Place one order at the beginning of January.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for January is 36 units, that for February is 60 units, and that for March is 85 units. If we place an order for 181 units at the beginning of January, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 181 units in full at the beginning of January, the inventory level at the beginning of January is 181 units. The inventory level at the end of January is 145 units. So, the average inventory level in January would be $= \frac{181 + 145}{2}$ =163 units. The inventory level at the beginning of February would be 145 units and at the end of February would be 85. So, the average inventory level in February would, therefore, be $= \frac{145 + 85}{2}$ =115 units. The inventory level at the beginning of March would be 85 units and at the end of March would be 0. So, the average inventory level in March would, therefore, be $= \frac{85 + 0}{2}$ =42.5 units. The total holding cost for the months of January, February, and March would, therefore, be $=(163 + 115 + 42.5) \times \$1.75 = \$560.875$
Total inventory cost	The total inventory cost is \$80 + \$560.875 = \$640.875
Per unit cost (PUC)	The total demand for January, February, and March is 181 units; the PUC is $= \frac{\$640.875}{181}$ = \$3.540

At this stage, we see that the PUC for order horizon January–February–March (3.540) is greater than the PUC for two-period order horizon of January–February (2.802) (see Fig 5.5). Therefore, the optimal order quantity is the sum of the order quantities for January and February, which is 96 units.

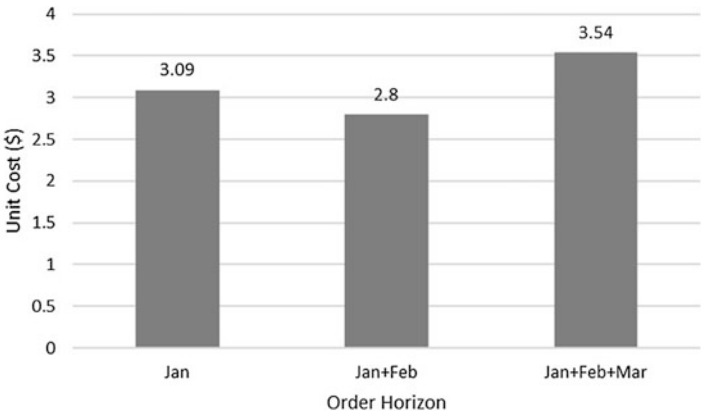

Fig. 5.5 Unit costs for January–February–March order horizon

Iteration 2.1

Next we set the order horizon to March because the demand for March was not included in the previous order.

Place an order for 85 units at the beginning of March.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for March is 85 units. If we place an order for 85 units at the beginning of March, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 85 units in full at the beginning of March, the inventory level at the beginning of March would be 85 units. The inventory level at the end of March is 0 units. So, the average inventory level in March would be $=\frac{85+0}{2}$ =42.5 units The total holding cost for March would be =(42.5) × \$1.75 = \$74.375
Total inventory cost	The total inventory cost is \$80 + \$74.375 = \$154.375
Per unit cost (PUC)	The total demand for March is 85 units. The PUC is $=\frac{\$154.375}{85}$ = \$1.816

Iteration 2.2
Next we set the order horizon to two periods – March and April.

Combine the demands for March and April and place one order at the beginning of March for 96 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The total demand for March and April is 96 units. If we place an order for 96 units at the beginning of March, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 96 units in full at the beginning of March, the inventory level at the beginning of March would be 96 units. The inventory level at the end of March is 11 units. So, the average inventory level in March would be $=\frac{96+11}{2}$ $=53.5$ units The total holding cost for March would be $=(53.5)\times\$1.75=\$93.625$ The inventory level at the beginning of April would be 11 units and at the end of April would be 0. So, the average inventory level in April would, therefore, be $=\frac{11+0}{2}$ $=5.5$ units The total holding cost for the months of March and April would, therefore, be $=(53.5+5.5)\times\$1.75=\$103.250$
Total inventory cost	The total inventory cost is \$80 + \$103.250 = \$183.250
Per unit cost (PUC)	Since this cost is incurred over two periods (March and April), the PUC is $=\frac{\$183.250}{96}$ $=\$1.908$

As can be seen from Fig. 5.6, since PUC for March and April is greater than PUC for March alone, we stop the current iteration. The strategy of placing an order to meet the demand for March alone is better than placing an order for the combined demand for March and April.

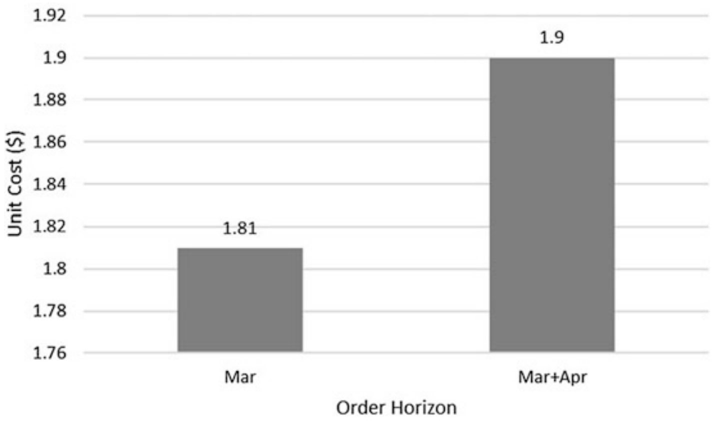

Fig. 5.6 Unit costs for March–April order horizon

Iteration 3.1
We start a new iteration by setting the start period to April.

Place one order at the beginning of April that meets its demand completely.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for April is 11 units. If we place an order for 11 units at the beginning of April, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 11 units in full at the beginning of April, the inventory level at the beginning of April is 11 units. The inventory level at the end of April is 0 units. So, the average inventory level in April would be $= \frac{11 + 0}{2}$ =5.5 units The total holding cost for the month of April would be $=(5.5) \times \$1.75 = \$9.625$
Total inventory cost	The total inventory cost is $\$80 + \$9.625 = \$89.625$
Per unit cost (PUC)	The total demand for April is 11 units, and the PUC is $= \frac{\$89.625}{11}$ $= \$8.147$

We next combine the demands for April and May (11 units +39 units = 50 units) to be placed at the beginning of April.

Iteration 3.2

We start the next iteration considering the demand for April and May.

Place one order at the beginning of April for 50 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

The demand for April is 11 units and May is 39 units. If we place an order for 50 units at the beginning of April, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 50 units in full at the beginning of April, the inventory level at the beginning of April would be 50 units. The inventory level at the end of April would be 39 units. So, the average inventory level in April would be $= \frac{50 + 39}{2}$ $=44.5$ units The inventory level at the beginning of May would be 39 units. The inventory level at the end of May would be 0 units. So, the average inventory level in May would be $= \frac{39 + 0}{2}$ $=19.5$ units The total holding cost for April and May would be $=(44.5 + 19.5) \times \$1.75 = \$112.00$
Total inventory cost	The total inventory cost is $\$80 + \$112 = \$192$
Per unit cost (PUC)	Since this cost is incurred over 50 units, the PUC is $= \frac{\$192}{50}$ $= \$3.84$

Since the PUC for the order horizon of April and May taken together is less than the PUC for the order horizon for April alone, we continue this iteration by combining the demand for April, May, and June in one order, placed at the beginning of April. The total demand for April, May, and June is 125 units.

Iteration 3.3

Combine the demands for April, May and June. Place one order at the beginning of April for 125 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 125 units at the beginning of April, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity of 125 units in full at the beginning of April, the inventory level at the beginning of April would be 125 units. The inventory level at the end of April would be 114 units. So, the average inventory level in April would be $= \frac{125 + 114}{2}$ $=119.5$ units The inventory level at the beginning of May would be 114 units and at the end of May would be 75. So, the average inventory level in May would, therefore, be $= \frac{114 + 75}{2}$ $=94.5$ units The inventory level at the beginning of June would be 75 units and at the end of June would be 0. So, the average inventory level in June would, therefore, be $= \frac{75 + 0}{2}$ $=37.5$ units The total holding cost for months of April, May, and June would, therefore, be $=(119.5 + 94.5 + 37.5) \times \$1.75 = \$440.125$
Total inventory cost	The total inventory cost is $\$80 + \$440.125 = \$520.125$
Per unit cost (PUC)	Since this cost is incurred over a total demand of 125 units, the PUC is $= \frac{\$520.125}{125}$ $= \$4.161$

As can be seen from Fig. 5.7, the PUC for the order horizon of April–May–June is greater than PUC for April–May. We stop the current iteration at this point.

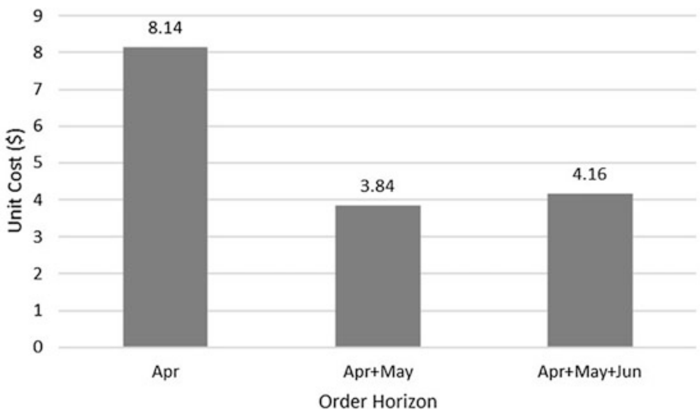

Fig. 5.7 Unit costs for April–May–June Order horizon

The strategy of placing an order to meet the demand for April–May is better than placing an order for the combined demand for April–May–June.

Iteration 4.1

We next start a new iteration after setting the starting period to June. The demand for June is 75 units. Since we currently do not know the demands for the months beyond June, we assume that it is economical to place an order for 75 units at the beginning of June.

Place one order at the beginning of June for 75 units.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (Kgs)	36	60	85	11	39	75

If we place an order for 75 units at the beginning of June, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity of 75 units in full at the beginning of June, the inventory level at the beginning of June would be 75 units. The inventory level at the end of June would be 0 units. So, the average inventory level in June would be $= \frac{75 + 0}{2}$ =37.5 units The total holding cost for month of June would, therefore, be $=(37.5) \times \$1.75 = \$65.625$
Total inventory cost	The total inventory cost is $\$80 + \$65.625 = \$145.625$
Per unit cost (PUC)	Since this cost is incurred over a total demand of 75 units, the PUC is $= \frac{\$145.625}{75}$ $= \$1.94$

We stop the solution process here. The least unit cost based solution summary shown in Table 5.9 suggests we place an order for 96 units in January (to cover

Table 5.9 Least unit cost solution summary

Month	Order quantity	Ordering cost	Holding cost	Inventory cost
January	96	\$80	\$189.00	\$269.00
February				
March	85	\$80	\$74.38	\$154.38
April	50	\$80	\$112.00	\$192.00
May				
June	75	\$80	\$65.62	\$145.62
Total inventory cost (TIC)		\$320	\$441.00	\$761.00

demand for January and February), 85 units in March, 50 units in April (to cover demand for April and May), and 75 in June.

The total inventory cost for the planning horizon is $761.

Solved Problem 5.4

Monthly demand for an item over 6 months is 32, 19, 12, 15, 23, and 12, units respectively. Using the least unit cost method, determine the total inventory cost if the holding cost is $1.5 per unit per month and the ordering cost is $40 per order.

Solution

Alternative	Order cost ($)	Holding cost ($)	Per unit cost	Remarks
Order 32 units to cover the demand for Month 1	40	24	2.00	Continue
Order 51 units to cover the demand for Month 1 and Month 2	40	66.75	2.09	Stop as unit cost is greater than the previous iteration
Since PUC for Month 1 is the least, the optimal lot size is 32 units for the order to be placed at the beginning of Month 1.				
Order 19 units to cover the demand for Month 2	40	14.25	2.86	Continue
Order 31 units to cover the demand for Month 2 and Month 3	40	41.25	2.62	Continue
Order 46 units to cover the demand for Month 2 through 4	40	45.83	2.99	Stop
Since PUC for Month 2 and Month 3 combined is the least, the optimal lot size is 31 units for the order to be placed at the beginning of Month 2.				
Order 15 units to cover the demand for Month 4	40	11.25	3.42	Continue
Order 38 units to cover the demand for Month 4 and Month 5	40	63	2.71	Continue
Order 50 units to cover the demand for Month 4 through Month 6	40	108	2.96	Stop
Since PUC for Month 4 through 5 is the least, the optimal lot size is 38 units for the order to be placed at the beginning of Month 4				
Order 12 units to cover the demand for Month 6	40	9.0	4.08	
Since we do not have further information, we stop the solution process assuming it is optimal to place an order in Month 6 for 12 units.				
Total Inventory Cost: 40 + 24 + 40 + 41.25 + 40 + 63.0 + 40 + 9.0 = $297.25				

5.2.5 *Wagner-Whitin Heuristic*

The Wagner-Whitin (Wagner and Whitin 1958) lot-sizing method is one of the more difficult methods to solve by hand. It uses a forward recursive algorithm. Using this algorithm, we first solve a one-period problem. We continue to sequentially solve subproblems until the solution is found for the entire planning horizon. It adopts the following principles:

- An order is placed at the beginning of the month.
- The order quantity is equal to the demand over a predecided order horizon.
- If some inventory exists at the beginning of the month, the inventory must be sufficient to meet the demand for a certain number of months.

Let us now learn the Wagner-Whitin algorithm by solving the corn flour additive inventory problem presented in the running example. We use the same cost parameters that have been used earlier, i.e.,

- Ordering cost C_o is \$80 per order
- Holding cost C_h is \$1.75 per unit per period

The demand for the problem is as shown below. The planning horizon is 6 months.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand (kg)	36	60	85	11	39	75

Iteration 1

Like in any of the previous methods, we start the first iteration by placing an order for 36 units that would entirely meet the demand for January, and January only. If we place an order for 36 units at the beginning of January, the following costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 36 units. The inventory level at the end of January would be 0. The average inventory level in January would, therefore, be $= \frac{36 + 0}{2}$ $=18$ units Thus, the holding cost for January would be $18 \times \$1.75$ or \$31.5.
Total inventory cost	The total inventory cost of placing an order for 36 units in January is therefore $\$80 + \$31.5 = \$111.5$.

Since there is no other option possible, we consider this to be the most optimal way to achieve the objective of meeting the demand for the month of January.

Iteration 2

Next, we consider the demand for 60 units in the month of February, in addition to the demand for 36 units in January. There are two options by which the demand for February can be met:

- Option 1: Place an order for combined demand for January and February (total 96 units) at the beginning of January.
- Option 2: Use the optimal method to meeting January demand (36 units) determined using iteration 1, and place a new order to meet February demand (60 units).

If we use Option 1, we incur the following costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 96 units. The inventory level at the end of January would be 60. The average inventory level in January would, therefore, be $= \frac{96 + 60}{2}$ =78 units The inventory level at the beginning of February would be 60 units, and at the end of February it would be 0. So, the average inventory level in February would, therefore, be $= \frac{60 + 0}{2}$ =30 units Thus, the holding cost for Option 1 would be $(78 + 30) \times \$1.75 = \$189$.
Total Inventory Cost	The total inventory cost for Option 1 would be $\$80 + \$189 = \$269$

Option 2 involves performing two steps:

- Use the best way to meet January demand. From Iteration 1, we know that the total inventory cost of the best way to meet the January demand of 36 units is \$111.5.
- Place a new order to meet February demand of 60 units.

The following costs would be incurred if we place a new order to meet this demand:

Type of cost	Description
Ordering cost	Since an order is placed in February for 60 units, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of February, the inventory level at the beginning of February would be 60 units. The inventory level at the end of February would be 0. The average inventory level in February would, therefore, be $= \frac{60 + 0}{2}$ =30 units Thus, the holding cost for February would be $(30) \times \$1.75$ or \$52.5.
Total inventory cost	The total inventory cost for Option 2 is $\$111.5 + \$80 + \$52.5 = \$244$

Table 5.10 Summary of total inventory costs – Iteration 2

Option	Remarks	Best possible cost	Holding cost for remaining Months	Ordering cost	Total cost
Option 1	Order to cover demand for January and February, in January	NA	$(108 \times 1.75) =$ $189.0	$80	$269.00
Option 2	Order to cover demand for January in best possible way; order for February in February	$111.00 (Iteration 1)	$(30 \times 1.75) =$ $52.5	$80	$244.00

Comparing the total inventory costs for Option 1 and Option 2, we see that the best way to meet the demand for January and February is to place an order separately for each month. The inventory costs for Iteration 2 have been summarized in Table 5.10.

Decision: From Table 5.10, we can see that it is optimal to use Option 2, which is to cover the demand for January in the best possible way, and then place an order for February in February.

Iteration 3

Next, we consider the demand of 85 units in the month of March. The demand for March can be met in three possible ways:

- Option 1: Place an order for combined demand for January, February, and March (total 181 units) at the beginning of January.
- Option 2: Use the best method to meet January demand (36 units), and place a new order at the beginning of February to meet demand for February and March (145 units).
- Option 3: Use the best method to meet February demand (60 units), and place a new order at the beginning of March to meet demand for March (85 units).

If we use Option 1, the following would be the total inventory costs:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 181 units. The inventory level at the end of January would be 145. The average inventory level in January would, therefore, be $= \frac{181 + 145}{2}$ $=163$ units The inventory level at the beginning of February would be 145 units and at the end of February would be 85. So, the average inventory level in February would, therefore, be $= \frac{145 + 85}{2}$ $=115$ units

(continued)

Type of cost	Description
	The inventory level at the beginning of March would be 85 units and at the end of March would be 0. So, the average inventory level in March would, therefore, be $= \frac{85 + 0}{2}$ =42.5 units Thus, the holding cost for Option 1 would be $(163 + 115 + 42.5) \times \$1.75 = \$560.875$.
Total inventory cost	The total inventory cost for Option 1 would be $\$80 + \$560.875 = \$640.875$

Option 2 involves performing two steps:

- Use the best way to meet January demand. From Iteration 1, we know that the total inventory cost for the best way to meet the January demand of 36 units is $111.5.
- Place a new order in February to meet demand for February and March (145 units).

The following costs would be incurred if we place a new order to meet this demand:

Type of cost	Description
Ordering cost	Since an order is placed in February for 145 units, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of February, the inventory level at the beginning of February would be 145 units. The inventory level at the end of February would be 85. The average inventory level in February would, therefore, be $= \frac{145 + 85}{2}$ =115 units The inventory level at beginning of March would be 85 units. The inventory level at the end of March would be 0. The average inventory level in March would, therefore, be $= \frac{85 + 0}{2}$ =42.5 units Thus, the holding cost for Option 2 would be $(115 + 42.5) \times \$1.75$ or $275.625
Total inventory cost	The total inventory cost for Option 2 is $\$111.5 + \$80 + \$275.625 = \$467.125$

Option 3 involves using the best way to meet the February demand (60 units), and place a new order at the beginning of March to meet demand for March (85 units).

Option 3 involves performing two steps:

- Use the best way to meet February demand. From Iteration 2, we know that the total inventory cost for the best way to meet the February demand is $244.
- Place a new order to meet March demand of 85 units. The following costs would be incurred if we place a new order to meet this demand:

Type of cost	Description
Ordering cost	Since an order is placed in March for 85 units, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of March, the inventory level at the beginning of March would be 85 units. The inventory level at the end of March would be 0. The average inventory level in March would, therefore, be $= \frac{85+0}{2}$ =42.5 units Thus, the holding cost for March would be (42.5) × $1.75 or $74.375.
Total inventory cost	The total inventory cost for Option 3 is $244 + $80 + $74.375 = $398.37

Comparing the total inventory costs for all the three options, we see that the best way to meet the demand for January through March is to place an order separately for each month, i.e., 36 units at the beginning of January, 60 units in the beginning of February, and 85 units in the beginning of March.

Decision: From Table 5.11, we can see that it is optimal to use Option 3, which is to cover the demand for February in the best possible way, and then place an order for March in March.

The inventory costs for Iteration 2 have been summarized in Table 5.11.

Iteration 4

Next, we consider the demand for 11 units in the month of April. The demand for April can be met in four possible ways:

- Option 1: Place an order for combined demand for January, February, March, and April (total 192 units) at the beginning of January.

Table 5.11 Summary of total inventory costs – Iteration 3

Option	Remarks	Best possible cost	Holding cost for remaining Months	Ordering cost	Total cost
Option 1	Order to cover demand for January to March, in January	NA	(320.5 × 1.75) = $560.88	$80	$640.88
Option 2	Order to cover demand for January in best possible way; order for remaining months in February	$111.5 (Iteration 1)	(157.5 × 1.75) = $275.63	$80	$467.13
Option 3	Order to cover demand for February in best possible way; order for remaining in March	$244.0 (Iteration 2)	(42.5 × 1.75) = $74.38	$80	$398.38

- Option 2: Use the best way to meet January demand (36 units), and place a new order at the beginning of February to meet the demand for February, March, and April (156 units).
- Option 3: Use the best way to meet February demand (60 units), and place a new order at the beginning of March to meet the demand for March and April (96 units).
- Option 4: Use the best way to meet March demand (85 units), and place a new order at the beginning of April to meet the demand for this month (11 units).

If we use option 1, the following inventory costs would be incurred:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of \$80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of January would be 192 units. The inventory level at the end of January would be 156. The average inventory level in January would, therefore, be $= \frac{192 + 156}{2}$ =174 units The inventory level at the beginning of February would be 156 units and that at the end of February would be 96. So, the average inventory level in February would, therefore, be $= \frac{156 + 96}{2}$ =126 units The inventory level at the beginning of March would be 96 units and that at the end of March would be 11. So, the average inventory level in March would, therefore, be $= \frac{96 + 11}{2}$ =53.5 units The inventory level at the beginning of April would be 11 units and at the end of April would be 0. So, the average inventory level in April would, therefore, be $= \frac{11 + 0}{2}$ =5.5 units Thus, the holding cost for Option 1 would be $(174 + 126 + 53.5 + 5.5) \times$ \$1.75 = \$628.25.
Total inventory cost	The total inventory cost for Option 1 would be \$80 + \$628.25 = \$708.25

Option 2 involves performing two steps:

- Use the best way to meet January demand. From Iteration 1, we know that the total inventory cost for the best way to meet the January demand of 36 units is \$111.5.
- Work to optimize the demand for the periods February through April (156 units). The following costs would be incurred if we place a new order to meet this demand for 156 units in the beginning of February:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of February would be 156 units and at the end of February would be 96. So, the average inventory level in February would, therefore, be $=\frac{156+96}{2}$ =126 units The inventory level at the beginning of March would be 96 units and at the end of March would be 11. So, the average inventory level in March would, therefore, be $=\frac{96+11}{2}$ =53.5 units The inventory level at the beginning of April would be 11 units and at the end of April would be 0. So, the average inventory level in April would, therefore, be $=\frac{11+0}{2}$ =5.5 units Thus, the holding cost for Option 1 would be $(126+53.5+5.5)\times\$1.75$ $=\$323.75$.
Total inventory cost	The total inventory cost for Option 2 would be $\$111.5+\$80+\$323.75=$ $515.25

Option 3 involves performing two steps:

- Use the best way to meet February demand. From Iteration 2, we know that the Total Inventory Cost for the best way to meet the February demand of 60 units is $244.
- Work to optimize the demand for the periods March through April (96 units).

The following costs would be incurred if we place a new order to meet this demand for 96 units in the beginning of February:

Type of cost	Description
Ordering cost	Since an order is placed, we incur an ordering cost of $80.
Holding cost	Since we receive the ordered quantity in full at the beginning of January, the inventory level at the beginning of March would be 96 units and at the end of March would be 11. So, the average inventory level in March would, therefore, be $=\frac{96+11}{2}$ =53.5 units The inventory level at the beginning of April would be 11 units and at the end of April would be 0. So, the average inventory level in April would, therefore, be $=\frac{11+0}{2}$ =5.5 units Thus, the holding cost for Option 3 would be $(53.5+5.5)\times\$1.75=\$103.25$.
Total Inventory Cost	The total inventory cost for Option 3 would be $\$244+\$80+\$103.25=$ $427.25

Option 4 involves performing two steps:

- Use the best way to meet March demand. From Iteration 1, we know that the Total Inventory Cost of the best way to meet the March demand of 85 units is $398.37.
- Work to optimize the demand for April (11 units).

The following costs would be incurred if we place a new order to meet this demand for 11 units in the beginning of April:

Type of Cost	Description
Ordering Cost	Since an order is placed, we incur an ordering cost of $80.
Holding Cost	Since we receive the ordered quantity in full at the beginning of April, the inventory level at the beginning of April would be 11 units and at the end of April would be 0. So, the average inventory level in April would, therefore, be $= \frac{11 + 0}{2}$ =5.5 units Thus, the holding cost for Option 4 would be (5.5) × $1.75 = $9.63.
Total Inventory Cost	The total inventory cost for Option 4 would be $398.37 + $80 + $9.63 = $488.00

The inventory costs for Iteration 4 have been summarized in Table 5.12.

Decision: From Table 5.12, we can see that it is optimal to use Option 3, which is to cover the demand for February in the best possible way, and then place an order for March and April in March.

Iteration 5

Next, we consider the demand for 39 units in the month of May. The demand for May can be met in five possible ways:

Table 5.12 Summary of total inventory costs – Iteration 4

Option	Remarks	Best possible cost	Holding cost for remaining Months	Ordering cost	Total cost
Option 1	Order to cover demand for January–May, in January	NA	(359 × 1.75) = $628.25	$80	$708.25
Option 2	Order to cover demand for January in best possible way; order for remaining months in February	$111.5 (Iteration 1)	(185 × 1.75) = $323.75	$80	$515.25
Option 3	Order to cover demand for February in best possible way; order for remaining months in March	$244.0 (Iteration 2)	(59 × 1.75) = $103.25	$80	$427.25
Option 4	Order to cover demand for March in best possible way; order for April in April	$398.37 (Iteration 3)	(5.5 × 1.75) = $9.63	$80	$488.00

Table 5.13 Summary of total inventory costs – Iteration 5

Option	Remarks	Best possible cost	Holding cost for remaining Months	Ordering cost	Total cost
Option 1	Order to cover demand for January–May, in January	NA	(534.5 × 1.75) = $935.38	$80	$1015.38
Option 2	Order to cover demand for January in best possible way; order for remaining months in February	$111.5 (Iteration 1)	(321.5 × 1.75) = $562.63	$80	$754.13
Option 3	Order to cover demand for February in best possible way; order for remaining months in March	$244.0 (Iteration 2)	(156.15 × 1.75) = $273.88	$80	$597.88
Option 4	Order to cover demand for March in best possible way; order for remaining months in April	$398.37 (Iteration 3)	(64 × 1.75) = $112	$80	$590.37
Option 5	Order to cover demand for April in best possible way; order for May, in May	$427.25 (Iteration 4)	(19.5 × 1.75) = $34.13	$80	$541.38

- Option 1: Place an order for combined demand for January, February, March, April, and May (total 231 units) at the beginning of January.
- Option 2: Use the best way to meet January demand (36 units), and place a new order at the beginning of February to meet the demand for February through May (195 units).
- Option 3: Use the best way to meet February demand (60 units), and place a new order at the beginning of March to meet the demand for March through May (135 units).
- Option 4: Use the best way to meet March demand (85 units), and place a new order at the beginning of April to meet the demand for April and May (50 units).
- Option 5: Use the best way to meet April demand (11 units), and place a new order at the beginning of May (39 units).

The inventory costs for Iteration 5 have been summarized in Table 5.13.

Decision: From Table 5.13 we can see that it is optimal to use Option 5, which is to place an order to cover the demand for May, in May.

Iteration 6

Next, we consider the demand of 75 units in the month of June. The demand for June can be met in six possible ways:

- Option 1: Place an order for combined demand for January through June (total 306 units) at the beginning of January.

- Option 2: Use the best way to meet January demand (36 units), and place a new order at the beginning of February to meet the demand for February through June (270 units).
- Option 3: Use the best way to meet February demand (60 units), and place a new order at the beginning of March to meet the demand for March–June (210 units).
- Option 4: Use the best way to meet March demand (85 units), and place a new order at the beginning of April to meet the demand for April–June (125 units).
- Option 5: Use the best way to meet April demand (11 units), and place a new order at the beginning of May to meet the demand for May–June (114 units).
- Option 6: Use the best way to meet May demand (39 units), and place a new order at the beginning of June to meet the demand for June (75 units).

The inventory costs for Iteration 6 have been summarized in Table 5.14

Decision: From Table 5.14, we can see that it is optimal to use Option 6, which is to place an order to cover the demand for June, in June. The solution summary to this lot-sizing problem based on the Wagner-Whitin method is as shown in Table 5.15.

The solution produced by the Wagner-Whitin method suggests that we place an order for 36 units in January, 60 units in February, 96 units in March (that would cover the demand for March and April), 39 units in May, and 75 in June. The total inventory cost for the planning horizon is $687.

Table 5.14 Summary of total inventory costs – Iteration 6

Option	Remarks	Best possible cost	Holding cost for remaining months	Ordering cost	Total cost
Option 1	Order to cover demand for January to June, in January	NA	$(947 \times 1.75) =$ $1657.25	$80	$1737.25
Option 2	Order to cover demand for January in best possible way; order for remaining months in February	$111.5 (Iteration 1)	$(621.5 \times 1.75) =$ $1087.63	$80	$1279.13
Option 3	Order to cover demand for February in best possible way; order for remaining months in March	$244.0 (Iteration 2)	$(381.5 \times 1.75) =$ $667.63	$80	$991.63
Option 4	Order to cover demand for March in best possible way;. Order for remaining months in April	$398.37 (Iteration 3)	$(214 \times 1.75) =$ $374.50	$80	$852.87
Option 5	Order to cover demand for April in best possible way; order for May and June, in May	$427.25 (Iteration 4)	$(132 \times 1.75) =$ $231.0	$80	$738.25
Option 6	Order to cover demand for April in best possible way; order for June, in June	$541.4 (Iteration 5)	$(37.5 \times 1.75) =$ $65.63	$80	$687.00

Table 5.15 Wagner-Whitin solution summary

Month	Order quantity	Ordering cost	Holding cost	Inventory cost
January	36	$80	$31.50	$111.50
February	60	$80	$52.50	$132.50
March	96	$80	$103.25	$183.25
April				
May	39	$80	$34.13	$114.13
June	75	$80	$65.62	$145.62
Total inventory cost (TIC)		$400	$287.00	$687.00

5.3 Summary

In this chapter, we explored the lot-sizing heuristics, methods that help find feasible solutions to practical lot-sizing issues when demand is known but varies by time. Each heuristic has its own unique way of approaching the solution.

The lot-for-lot method requires very little computation. It states that we place an order for a quantity in a period that completely meets the demand for that period. The other heuristics require some amount of computation. The part-period balancing method tries to find a feasible solution by balancing the holding cost with the ordering cost. The Silver-Meal and least unit cost methods produce a feasible solution by using the concept of average inventory cost – by period and by quantity (or units) – respectively.

Wagner-Whitin method possibly requires most calculations as it uses a forward recursive algorithm to sequentially solving the lot-sizing problem. All these methods, however, require that the holding costs and ordering costs are estimated accurately.

There is no such thing as the best method, and depending on the situation one of these methods can be implemented to obtain least cost inventory management solution.

5.4 Case Study – Finishing School for Investment Bankers

Promantia LLP is a training company that focuses on Investment Banking professionals. They conduct several introductory and advanced short-term courses in Investment Banking. Their flagship course is entitled Equities and Bonds. This course is targeted at Investment Banking professionals with 0–2 years' experience in the financial industry. Most investment Banks regularly send their entry level new hires (executives) to Promantia to attend this short-term (10-day) course to get them introduced to financial and investment business.

Training Calendar

Anita is a Sales Manager with Promantia. Each year in November and May, Anita meets up with her Investment Banking clients to understand their future training needs. She collates this information and prepares a firm training calendar for the next 6 months. To minimize organizing costs, Promantia conducts only one training course of a type each month.

In November 2016, Anita has finalized the training calendar for the first half of 2017. Table below shows, by month, the expected number of executives that would be attending the short-term course on Equities and Bonds.

Month	Jan	Feb	Mar	Apr	May	Jun
Expected attendees	18	12	16	19	21	11

From the table, Promantia now knows that 18 executives would attend the training course in January, 12 executives would attend in February, 16 in March, and so on.

Ordering Course Material

Ismail is an Administrative Assistant with Promantia. His primary task is to make sure that executives attending the training course are provided with a standard course material (training material). He uses the training calendar published by Anita to perform his job.

Ismail gets the training material printed from Printo, a large 24 x 7 digital printing chain. Printo takes less than 1 h to print, bind, and deliver the training material in a ready-to-use form – therefore, the lead time may be considered negligible.

Ismail uses a Lot-for-Lot ordering policy to order training materials. One day before the scheduled training course Ismail places an order with Printo and gets the student training material delivered by end of the day. For example, if the training for the month of January is scheduled to start on the January 9, 2017, he would place an order on the morning of January 8 and get the material by 6:00 pm the same day. This way he makes sure that the executives attending the training course have their material at the start of the training course.

Drive to Optimize

You are an Industrial Engineering Consultant. Promantia has approached you to advise them on optimizing their processes, and to begin with they have requested you to focus on inventory management of training material. They have provided you with the demand for training materials for their flagship course Equities and Bonds. Based on your interaction with staff at Promantia you have estimated that the holding cost is \$2.5 per item per month and the ordering costs is \$60 per order.

Using the information provided, answer the following questions with respect to managing the inventory of Promantia's flagship course Equities and Bonds:

(a) Compute the total inventory costs for the current situation where Ismail is following a Lot-for-Lot ordering policy to order training materials (Answer shown in Table 5.16).
Answer:

Table 5.16 Lot-for-Lot Ordering policy solution for Promantia LLP case study

Alternative	Order cost ($)	Holding cost ($)	Total cost ($)
Order 18 to cover the demand for January	60	22.5	82.5
Order 12 to cover the demand for February	60	15	75
Order 16 to cover the demand for march	60	20	80
Order 19 to cover the demand for April	60	23.75	83.75
Order 21 to cover the demand for may	60	26.25	86.25
Order 11 to cover the demand for June	60	13.75	71.75
Total inventory cost = $481			481

Table 5.17 PPB solution for Promantia LLP case study

Alternative	Order cost ($)	Holding cost ($)	Closeness factor	Remarks
Order 18 to cover the demand for January	60	22.5	37.5	Continue until holding costs pass order cost
Order 30 to cover the demand for January and February	60	67.5	7.5	Stop as holding costs are larger than order cost
Since closeness factor for 30 units is smaller (i.e., holding cost is closer to order cost) than that for 18 units, the optimal lot size is 30 units, for the order to be placed at the beginning of January.				
Order 16 units to cover the demand for March	60	20	40	Continue until holding costs pass order cost
Order 35 units to cover the demand for March and April	60	91.25	31.25	Stop as holding costs are larger than order cost
Since closeness ratio for 35 units is smaller than that for 16 units, the optimal lot size is 35 units, for the order to be placed at the beginning of March.				
Order 21 to cover the demand for May	60	26.25	33.75	Continue until holding costs pass order cost
Order 32 to cover the demand for May and June	60	67.5	7.5	Stop as holding costs are larger than order cost
Since closeness ratio for 32 units is smaller than that for 21 units, the optimal lot size is 32 units, for the order to be placed at the beginning of May.				
Total Inventory Cost: 60 + 67.5 + 60 + 91.25 + 60 + 67.5 = $406.25				

(b) Compute the total inventory costs if you were to use the Part-Period Balancing Heuristic (Answer shown in Table 5.17).
Answer:

(c) Compute the total inventory costs if you were to use the Silver-Meal Heuristic (Answer shown in Table 5.18).
Answer:

(d) Compute the total inventory costs if you were to use the least unit cost heuristic (Answer shown in Table 5.19).
Answer:

Table 5.18 Silver-Meal solution for Promantia LLP case study

Alternative	Order cost ($)	Holding cost ($)	Per period cost	Remarks
Order 18 to cover the demand for January	60	22.5	82.5	Continue
Order 30 to cover the demand for January and February	60	67.5	63.75	Since PPC is lower than the previous iteration, continue batching demand
Order 46 to cover the demand for January February, and March	60	167.5	75.83	Stop as PPC is greater than the previous iteration
Since PPC for January and February combination is the least, the optimal lot size is 30 units, for the order to be placed at the beginning of January.				
Order 16 units to cover the demand for March	60	20	80	Continue
Order 35 units to cover the demand for March and April	60	91.25	75.63	Since PPC is lower than the previous iteration, continue batching demand
Order 56 units to cover the demand for March, April and May	60	222.5	94.17	Stop as PPC is greater than the previous iteration
Since PPC for March and April is the least, the optimal lot size is 35 units, for the order to be placed at the beginning of March.				
Order 21 to cover the demand for May	60	26.25	86.25	Continue
Order 32 to cover the demand for May and June	60	67.5	63.75	Solution can stop here since we do not have information about the next period
Since PPC for May and June is the least, the optimal lot size is 32 units, for the order to be placed at the beginning of May.				
Total Inventory Cost: 60 + 67.5 + 60 + 91.25 + 60 + 67.5 = $406.25				

5.5 Practice Problems

Problem 5.1

Consider an item whose annual demand is 120 units, the holding cost is $8 per item per annum ($0.67 per item per month), and the ordering cost is $19.2 per order. If the demand varies by month as shown in the table below, determine the total

Table 5.19 Least unit cost solution for Promantia LLP case study

Alternative	Order cost ($)	Holding cost ($)	Per Unit cost	Remarks
Order 18 to cover the demand for January	60	22.5	4.58	Continue
Order 30 to cover the demand for January and February	60	67.5	4.25	Since unit cost is lower than the previous iteration, continue batching demand
Order 46 to cover the demand for January, February, and March	60	167.5	4.94	Stop as unit cost is greater than the previous iteration
Since PUC for January and February combination is the least, the optimal lot size is 30 units, for the order to be placed at the beginning of January.				
Order 16 units to cover the demand for March	60	20	5.0	Continue
Order 35 units to cover the demand for March and April	60	91.25	4.32	Since unit cost is lower than the previous iteration, continue batching demand
Order 56 units to cover the demand for March, April, and May	60	222.5	5.04	Stop as unit cost is greater than the previous iteration
Since PUC for March and April is the least, the optimal lot size is 35 units, for the order to be placed at the beginning of March.				
Order 21 to cover the demand for May	60	26.25	4.11	Continue
Order 32 to cover the demand for May and June	60	67.5	3.98	Solution can stop here since we do not have information about the next period
Since PUC for May and June is the least, the optimal lot size is 32 units, for the order to be placed at the beginning of May				
Total Inventory cost: 60 + 67.5 + 60 + 91.25 + 60 + 67.5 = $406.25				

inventory cost if you would use the standard Economic Order Quantity (EOQ) policy[3] to manage your inventory. Assume instantaneous replenishment.

Period	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Demand	10	12	2	18	5	19	3	2	18	7	20	4

Hint

First compute the EOQ using $Q^* = \sqrt{\frac{2DC_o}{C_h}}$ where D is the annual demand (120), C_o is the ordering cost per order ($19.2), and C_h is the holding cost ($8 per item per annum). Substituting these values, we get Q^*=24 units.

[3]It should be noted that EOQ must be used only when the demand is constant, i.e., the demand does not vary by time. This question is provided only for student learning. For further reading please, see works by Nahmias (2015, pp. 459) and Srinivasan (2010, pp. 178).

Next, compute the reorder interval given by $T = \frac{Q^*}{D}$. Using the values of Q^* (24) and D (120), we get $T = 0.2$ years. If we assume 360 days in a year we get T = 72 days. This means we place an order of 24 units every 72 days. Thus, if the first order is placed on 1st January, the second order would be placed on 12th March, the third order on 24th May, the fourth order on 6th August, the fifth order on 18th October, and the last order on 31st December.

Using the above information, we can calculate the beginning and ending inventories for each month, as shown below:

Period	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Demand	10	12	2	18	5	19	3	2	18	7	20	4
Order receipt	24		24		24			24		24		24
Beginning inventory	24	14	1	25	7	26	7	4	26	8	25	5
Ending inventory	14	2	25	7	26	7	4	26	8	25	5	25

Answer:

Number of orders: 6, so total ordering cost $= 6 \times 19.2 = \$115.20$
Inventory holding cost = \$115.33
Total Inventory Costs = \$115.20 + \$115.33 = \$230.53

Problem 5.2

Consider the situation in problem 5.1. If you were to use Lot-for-Lot policy, what would your inventory costs be?

Answer:

Number of orders: 12, so total ordering cost $= 12 \times 19.2 = \$230.40$
Inventory holding cost = \$40.00
Total Inventory Costs = \$230.4 + \$40 = \$270.40

Problem 5.3

Karla, an Inventory Manager of a large retail warehouse, has done considerable research in application of part-period balancing lot-sizing heuristic. This heuristic tries to balance the holding cost for an order horizon with the ordering cost. Mathematically, if the total holding cost an order horizon of $(j-1)$ periods is closer to the ordering cost, then the part-period balancing heuristic sets the optimal order horizon to $(j-1)$ periods. The optimal order quantity in such a case is given by

$$Q_{j-1} = \sum_{i=1}^{j-1} D_i$$

where, $D_1, D_2, D_3, \cdots, D_n$ is the demand for an item over a horizon of n periods, and $i \in j$.

However, Karla believes that the closeness of the holding cost to the ordering cost must be determined based on the ratio rather that absolute numbers. Mathematically, Karla believes that to determine the closeness one need to find the ratio of the ordering cost to the holding cost for an order horizon, given by

$$C_{\mathrm{r}} = \frac{C_{\mathrm{o}}}{C_{\mathrm{h}} \sum_{i=1}^{j} \frac{(2i-1)D_i}{2}}$$

If the C_r over an order horizon of $(j-1)$ periods is closer to 1 than that of j periods, then we set the optimal order horizon to $(j-1)$ periods. Here, C_o is the ordering cost and C_h is the holding cost per item per period.

If you were to use Karla's suggestion to solve Rosetta's problem presented in the running example, what would be the total inventory cost? Use the same parameters, i.e., Order cost $C_o$is $80 per order and Holding cost $C_h$is $1.75 per unit per period, and compare the results [Level: Hard].

Answer:

If we were to use Karla's suggestion of using a ratio rather than absolute numbers we would incur the following inventory costs:

Month	Order quantity	Order cost	Holding cost	Inventory cost
January	96	$80	$189.00	$269.00
February				
March	85	$80	$74.38	$154.38
April	50	$80	$112.00	$192.00
May				
June	75	$80	$65.63	$145.63
Total cost (TIC)		$320	$441.01	$761.01

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Chapter 6
Stochastic Inventory Models

6.1 Introduction

At Rosetta's, demand for vegetable oil has been fairly steady and constant, and that for corn flour additive has been predictable. But the demand for eggs has been very uncertain. There have been instances when less than 200 eggs have been consumed in a week. There have also been instances when more than 900 eggs have been used up in a week. On an average, the demand for eggs has been around 500 per week.
Because the demand for the item is not constant, the EOQ formula cannot be applied. How, then, would managers at Rosetta's manage inventory of items whose demand is uncertain? This concern will be addressed as part of this chapter.

In Chap. 3, we presented mathematical models that help us determine economic order quantity, Q, and reorder level, s, for continuous review-based systems, and the optimal time between review periods, T, for periodic review-based systems. One assumption we made was the demand and replenishment lead times were known with certainty. However, in real life this is not always true and, therefore, deterministic inventory models may not be suitable in all situations. Several methods have been developed to manage inventories in situations where demand and/or replenishment lead times are uncertain. Fig. 6.1 shows a high-level classification of these methods. Each of these methods is described in detail in this chapter.

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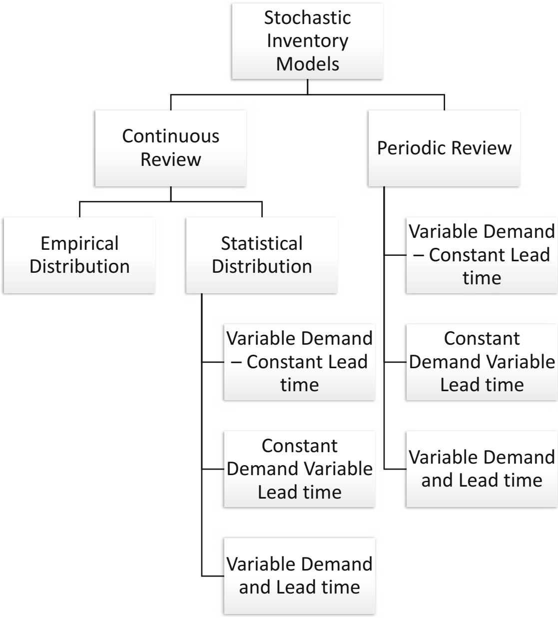
Fig. 6.1 Classification of stochastic inventory control models



6.2 Continuous Review-Based Models

It is recalled from Chap. 2 that in a continuous review-based inventory system a replenishment order is placed when the inventory falls to s, the reorder level, which is a function of lead time demand. When the demand is variable, a range of demands is possible during the lead time. Because of this, there is a possibility of stockout – a situation in which demand occurs and there is no inventory on hand to satisfy it. Every time a stockout occurs a cost is incurred. This cost is referred to as the stockout cost. Development of an inventory model involves balancing all types of inventory costs such as the carrying costs, ordering costs, and, in this situation, stockout costs. Computing stockout costs accurately[1] is extremely difficult (Monks 1987). Most researchers recommend maintenance of safety stock to reduce possibility of stockout.

6.3 Service Levels and Safety Stock

Safety stock (SS) is an additional amount of inventory that is carried to meet fluctuations in demand. Safety stock helps reduce the probability of stockout. A key pre-requisite though is the establishment of a service level. A service level is a policy measure set by inventory managers that help determine the level of safety

[1] Monks (1987) has suggested a simple model to compute Q when shortage costs are known. This is presented in Appendix 6A

Table 6.1 Service level calculation

Order cycle	Demand (in units)	Number of items out of stock
1	23	0
2	18	0
3	31	6
4	24	0
5	12	0
6	32	7
7	38	13
8	40	15
9	13	0
10	20	0
11	24	0
12	25	0
Total	300	41

Adapted from Nahmias (2005)

stock that needs to be maintained to protect themselves from stockout situations. Silver et al. (1998) have discussed several types of service levels in inventory management. Two types of service levels commonly used in inventory management are (1) a measure based on the proportion of order cycles in which no stockouts occur, called the *cycle service level* and (2) a measure based on the proportion of customer demands that are satisfied from the inventory on hand, also referred to as the *fill rate* (Nahmias 2005). Table 6.1 shows sales data for an item – demand in number of units and number of items out of stock – by order cycle.

Using the data presented, two types of service levels can be computed:

(a) The proportion of order cycles in which no stockouts occur, or *cycle service level*. No stockout was experienced in 8 of the 12 order cycles. The cycle service level, therefore, is $\frac{8}{12} = 0.67$. In other words, there is a 67% probability of not stocking out in a replenishment cycle.
(b) The proportion of demand that is met out of stock on hand, called the *fill rate*, is another service level which measures the stock performance. In the situation presented in Table 6.1, demand for 259 units was met from inventory held on hand, or fill rate is $\frac{(300-41)}{300} = 0.86$. In other words, the *fill rate* is 86%.

In the remainder of this chapter, we discuss various methods of calculation of safety stock for a specified type of service level.

6.4 Determining Safety Stock Level

There are several methods that can be used to determine safety stock. These methods require a thorough analysis of the historical demand and lead time data. Organizations may not always have sufficient data to analyze the variability of

Table 6.2 Running example – weekly demand for eggs

Weekly demand	Observed frequency	Weekly demand	Observed frequency
1–100	2	500–600	23
100–200	7	600–700	18
200–300	12	700–800	12
300–400	18	800–900	7
400–500	23	900–1000	2

demand during lead time. In such situations, they may use a frequency distribution to determine the level of safety stock.

6.4.1 Using Frequency and Cumulative Distribution

Consider an example of weekly demand for eggs at Rosetta's as shown in Table 6.2. The frequency column in the table shows the number of times the corresponding demand has been observed. For example, a demand of more than 100 eggs but less than 200 eggs has been observed 7 times out of a total 124 observations. Let us assume the procurement lead time for eggs from the local supplier is constant for 1 week.

The first step is to generate a cumulative probability distribution (see Table 6.3). The cumulative probability can be obtained by dividing the cumulative frequency by the total number of observations (124 observations in this case). For example, the probability that the demand would be up to 200 units is $\frac{9}{124}$ or 0.07. It also means that the probability that the demand would exceed 200 units is 0.93. The probability of demand fits in well with the concept of service level.

The next step is to plot a graph of demand (X-axis) vs service level (Y-axis). One may use MS Excel to generate a scatterplot, a graph that illustrates the relationship between two variables.[2] This is shown in Fig. 6.2. Once the graph is plotted, we can use the median, or the 50th percentile, to estimate the average weekly demand. From Fig. 6.2, we can see that the demand that corresponds to the median (or the 50th percentile) is 500. The average weekly demand D_a is, therefore, 500 eggs.

[2]In MS Excel 2016, Scatter Chart option is available from the Insert tab

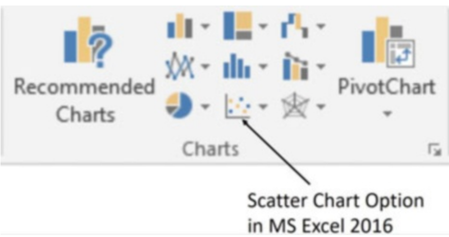


Table 6.3 Cumulative probability distribution

Weekly demand	Frequency	Cumulative frequency	Cumulative probability
1–100	2	2	0.02
100–200	7	9	0.07
200–300	12	21	0.17
300–400	18	39	0.31
400–500	23	62	0.50
500–600	23	85	0.69
600–700	18	103	0.83
700–800	12	115	0.93
800–900	7	122	0.98
900–1000	2	124	1.00

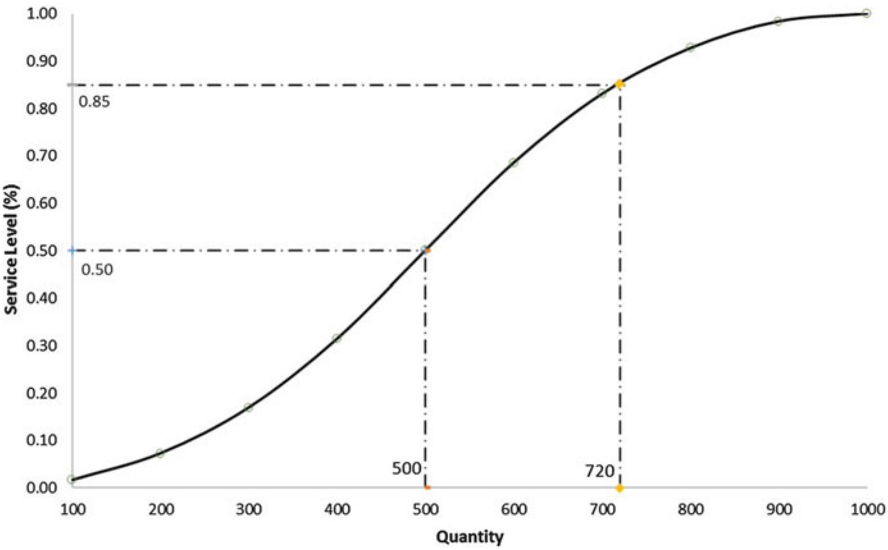


Fig. 6.2 Cumulative probability plot

We can also use the graph to estimate the demand for a service level of, say, 85%, which corresponds to the 85th percentile value of the cumulative distribution. As can be seen from the graph, this is 720. This means that there is an 85% chance that the demand for eggs in a given week may be up to 720. Alternatively, there is a 15% chance that the demand for eggs in a given week may exceed 720. We can now use this information to compute the level of safety stock. The safety stock (SS_{sl}) is the difference between the maximum demand established for a given service level (D_{sl}) and the average demand $\left(\bar{d}\right)$. Mathematically, this can be expressed as follows:

$$SS_{sl} = D_{sl} - \bar{d} \tag{6.1}$$

Since we know that the maximum demand for eggs for a service level of 85% is 720 and the average demand is 500, using Eq. 6.1, we have

$$SS_{0.85} = 720 - 500 = 220 \text{ eggs}$$

Thus, the safety stock required to maintain a service level of 85% is 220 eggs. The cost of maintaining the safety stock is a function of the inventory carrying rate (i), the cost of the item (C), and the size of the safety stock. This can be expressed as

$$C_{ss} = iC.SS_{sl} \tag{6.2}$$

where SS_{sl} is the safety stock for a desired service level.

The problem of variation in lead times can also be solved in a similar fashion by developing a cumulative probability distribution for lead times. If the service level is established, we can then find the safety stock, SS_{sl}, for the specified service level. It should be noted that in the case of variable lead time the safety stock would be expressed in terms of usage of inventory over a given period. Safety stock can be determined using Eq. 6.3:

$$SS_{sl} = LT_{sl} - \bar{L} \tag{6.3}$$

In this equation, LT_{sl} is the lead time for the specified service level and $\bar{L}$ is the average lead time (corresponding to 50th percentile on the cumulative probability plot).

Solved Problem 6.1

Historical data presented in Table 6.4 shows a manufacturer's weekly demand for an item that has a constant lead time of 1 week. The item costs $50 per unit. The manufacturer uses an inventory carrying rate of 20% per year. Determine the safety stock and the carrying costs if the manufacturer desires service levels of (a) 85% and (b) 95%.

Solution

This problem can be solved in the following four steps:

Step 1: First, we generate a cumulative probability distribution. This is shown in Table 6.5.

Step 2: We generate a plot of cumulative probability distribution, as shown in Fig. 6.3, with quantity on the X-axis and cumulative probability on the Y-axis.

Table 6.4 Data for solved Problem 6.1

Weekly demand	Frequency	Weekly demand	Frequency
1–50	2	250–300	9
50–100	4	300–350	7
100–150	7	350–400	6
150–200	8	400–450	5
200–250	9	450–500	3

Table 6.5 Cumulative probability distribution for solved Problem 6.1

Weekly demand	Frequency	Cumulative frequency	Cumulative probability
1–50	2	2	0.03
50–100	4	6	0.10
100–150	7	13	0.22
150–200	8	21	0.35
200–250	9	30	0.50
250–300	9	39	0.65
300–350	7	46	0.77
350–400	6	52	0.87
400–450	5	57	0.95
450–500	3	60	1.00
Total	60		

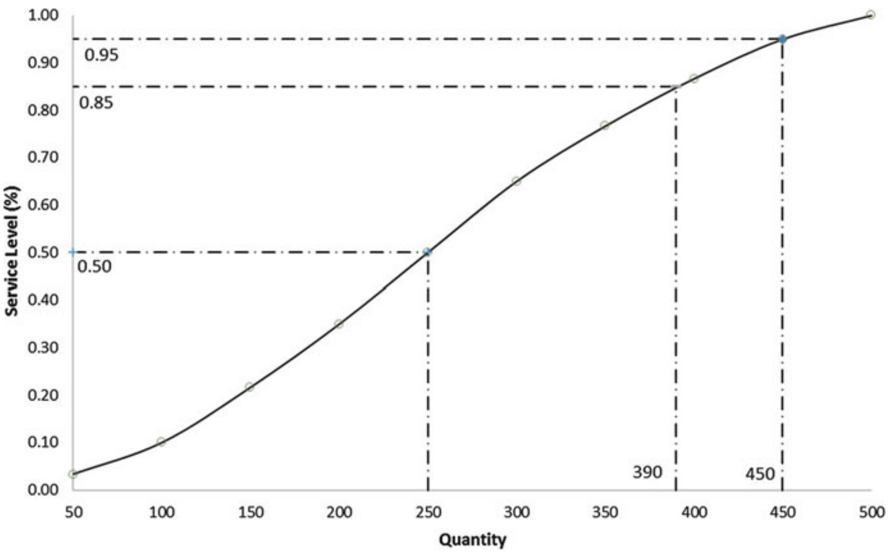


Fig. 6.3 Cumulative probability plot for solved Problem 6.1

From Fig. 6.3, we see that the demand that corresponds to the 50th percentile is 250. The average weekly demand $\bar{d}$ is, therefore, 250 units. We can also use the graph to estimate the demand for service levels of 85% and 95%, which corresponds to the 85th and 95th percentile value of the cumulative distribution. As can be seen from the graph, this is 390 and 450, respectively. Since the maximum demand for a service level of 85% ($D_{0.85}$) is 390 units and the average demand is 250, using Eq. 6.1, we have

$$SS_{0.85} = 390 - 250 = 140 \text{ units}$$

Table 6.6 Lead time observations for solved Problem 6.2

Lead time (days)	Frequency (Number of times this lead time has been observed)	Lead time (days)	Frequency (Number of times this lead time has been observed)
1	2	6	7
2	3	7	5
3	5	8	3
4	6	9	2
5	6	10	1

Thus, the safety stock required to maintain a service level of 85% is 140 units. The cost of maintaining safety stock can be computed using Eq. 6.2. The cost of maintaining this safety stock of 140 units is $0.2 \times 50 \times 140 = \$1{,}400$ per year. Similarly, the maximum demand for a service level of 95% is 450 units. Using Eq. 6.1, we have

$$SS_{0.95} = 450 - 250 = 200 \text{ units}$$

Thus, the safety stock required to maintain a service level of 95% is 200 units. The cost of maintaining this safety stock of 200 units is $0.2 \times 50 \times 200 = \$2000$ per year.

Solved Problem 6.2
A manufacturing firm procures raw materials from an international supplier. The procurement lead time varies as seen from the 40 observations in Table 6.6. What amount of safety stock (in days) would the manufacturer need to carry in order to ensure that they have the raw material on hand 85% of the time to meet the production requirements?

Solution
As seen earlier, this can be solved in the following three steps:

Step 1: Generate cumulative probability distribution as shown in Table 6.7.
Step 2: Generate a plot of cumulative probability distribution with lead time on the *X*-axis and cumulative probability on the *Y*-axis. This is as shown in Fig. 6.4.
Step 3: As can be seen from Fig. 6.4, the average lead time (corresponding to 50th percentile) is 4.7 days. The lead time that corresponds to 85th percentile is 7 days. Substituting the values for the average and 85th percentile lead times in Eq. 6.3, we get

$$SS_{sl} = 7 - 4.7 = 2.3 \text{ days}$$

The safety stock that is needed to maintain a service level of 85% is equal to 2.3 days' usage. If the daily usage of the raw material is known, the safety stock, in terms of units, can be easily determined.

Table 6.7 Cumulative probability distribution for solved Problem 6.2

Observed lead time (days)	Frequency	Cumulative frequency	Cumulative probability
1	2	2	0.05
2	3	5	0.13
3	5	10	0.25
4	6	16	0.40
5	6	22	0.55
6	7	29	0.73
7	5	34	0.85
8	3	37	0.93
9	2	39	0.98
10	1	40	1.00
Total	40		

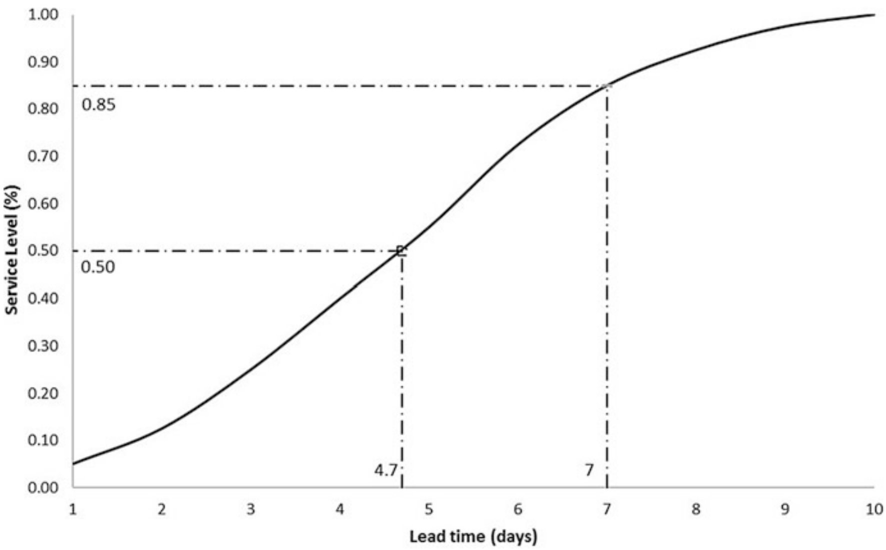


Fig. 6.4 Cumulative probability plot for solved Problem 6.2

6.4.2 Using Statistical Distributions

In the previous section, we discussed a method of determining the safety stock when the amount of available historical data is not very large. However, when sufficient amount of data is available, we may be in a position to fit it to a known statistical distribution.[3] When the lead time demand is variable, a range of demands is possible as shown in Fig. 6.5. In this section, we assume that the range of demand

[3]Software such as Minitab can be used to identify the distribution that fits the data, along with parameters.

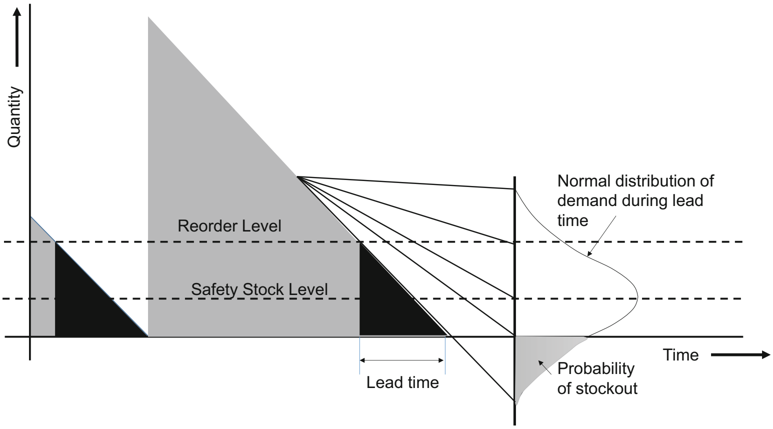


Fig. 6.5 Demand during lead time is normally distributed

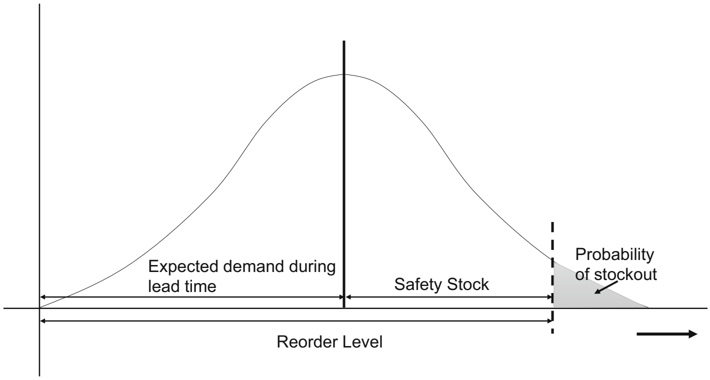


Fig. 6.6 Safety stock and probability of stockout

during lead time can be represented by a normal distribution, although other continuous distributions may also be used.

Figure 6.6 shows the relationship between the reorder level and safety stock for the desired service level. In the following sections, we describe methods for computing the reorder level as well as the safety stock for the following scenarios (Sharma 2006; Venkataraman and Pinto 2017; Sen 2012):

- Demand is variable, lead time is constant
- Lead time is variable, demand is constant
- Both demand and lead times are variable

6.4.2.1 Reorder Level – Variable Demand, Constant Lead Time

Let us first consider the case when demand is variable and the lead time is constant. Following assumptions have been made in determining the reorder level:

- The inventory system is reviewed continuously.
- The inventory system involves a single item.
- Demand is for the item is random, but the distribution governing the demand is known or can be estimated. We assume the demand is normally distributed with a known mean and standard deviation.
- Lead time is known and constant.
- A fixed setup cost is incurred every time an order is placed.

The order size can be determined using Eq. 3.8 as discussed in Chap. 3, reproduced as follows for convenience:

$$Q = \sqrt{\frac{2DC_{\mathrm{o}}}{iC}}$$

When demand is normally distributed with a known mean and standard deviation, the reorder level, s, is given by

$$s = \bar{d}L + z\sigma_{\mathrm{d}} \tag{6.4}$$

where

- $\bar{d}$ is the average demand;
- L is the lead time;
- z is the number of standard deviations for a specified cycle service level;
- σ_{d} is the standard deviation of *lead time demand*.

The units for average demand and lead time must be consistent, i.e., if the demand period is specified in days, then the lead time must also be in days. If the problem specifies standard deviation of *daily demand*, then Eq. 6.4 may also be written as

$$s = \bar{d}L + z\sigma_{\mathrm{i}}\sqrt{L} \tag{6.5}$$

where

- $\bar{d}$ is the average daily demand;
- L is the lead time in days;
- z is the number of standard deviations for a specified cycle service level;
- σ_{i} is the standard deviation of the *daily demand*.

Normal distribution functions in MS Excel

If the desired cycle service level is specified, the MS Excel function NORM . S . INV(p) may be used to obtain the value of z, the standard normal variate for the specified service level (or probability). For example, if the desired cycle service level is 85%, the MS Excel function returns:

$$z = \text{NORM.S.INV}(0.85) = 1.03$$

Another MS Excel function NORM.DIST can be used to obtain the cycle service level for a given replenishment policy. This function needs other inputs such as the reorder level (s), the lead time demand $\left(\bar{d}L\right)$, and the standard deviation of lead time demand (σ_d). The syntax for usage of the function is as follows:

$$\text{Cycle Service Level} = \text{NORM.DIST}\left(\text{s}, \bar{d}L, \sigma_\text{d}, \text{FLAG}\right)$$

For example, if the reorder level is 1000, lead time demand is 500 units, and the standard deviation of lead time demand is 300 units, MS Excel function returns:

$$= \text{NORM.DIST}(1000, 500, 300, \text{TRUE}) = 0.95$$

or the cycle service level is 95%. It should be noted that the FLAG must be always set to TRUE to obtain a cumulative value.

To determine the fill rate, f_r, we need to perform the following steps:

First, determine the expected shortage per order cycle, E_s. This can be estimated using the following function in MS Excel (Chopra and Meindl 2010):

$$E_\text{s} = \sigma \times \text{NORMDIST}\left(\frac{\text{SS}}{\sigma}, 0, 1, 0\right) - \text{SS}\left[1 - \text{NORMDIST}\left(\frac{\text{SS}}{\sigma}, 0, 1, 1\right)\right]$$

where

SS is the safety stock;
σ is the standard deviation of the lead time demand.

For example, if the safety stock is 75 units and the standard deviation of lead time demand is 12 units, then the expected shortage is

$$E_\text{s} = 60 \times \text{NORMDIST}\left(\frac{75}{60}, 0, 1, 0\right) - 75\left[1 - \text{NORMDIST}\left(\frac{75}{60}, 0, 1, 1\right)\right]$$
$$E_\text{s} = 3.0$$

(continued)

Once we have the expected shortage per order cycle, we can use the following equation to determine the fill rate:

$$f_{\mathrm{r}} = \frac{Q - E_{\mathrm{s}}}{Q}$$

For example, if the order quantity Q is 500 units and the expected shortage per cycle is 3 units, then the fill rate is

$$f_{\mathrm{r}} = \frac{500 - 3}{500} = 0.994$$

or the fill rate is 99.4%.

Demand period and lead time – things to consider

Demand period is the time period for which demand is specified. It is very important that the units used for all parameters in a given problem (such as demand period, lead time, etc.) be consistent. To solve a numerical problem, we would need the standard deviation of the lead time demand. Problems in inventory management may specify the lead time demand and standard deviation directly, which can be substituted in Eq. 6.4 to obtain the reorder level. However, in some cases, the problem may specify the standard deviation of the daily demand, σ_i. In this case, we may use Eq. 6.5 to solve the problem. If not, following adjustments will be required to be made.

Case 1: If demand period is less than lead time

Consider the following example where the standard deviation of *daily demand* is 4 units and the lead time is 3 days. Assuming the demand for each day is independent, the standard deviation of lead time demand is equal to the square root of the sum of the variances of daily demand (Jacobs and Chase 2011). In other words,

$$\sigma_{\mathrm{d}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \tag{6.6}$$

where

σ_{d} is the standard deviation of the *lead time demand*, and
$\sigma_{1\ldots3}$ is the standard deviation of the individual *daily demands*.

Substituting the values in Eq. 6.6, we get

$$\sigma_{\mathrm{d}} = \sqrt{4^2 + 4^2 + 4^2} = \sqrt{48} = 6.92 \text{ units}$$

The standard deviation of the lead time demand is 6.92 units.

Table 6.8 Decision rules to compute standard deviation of lead time demand

If demand period is smaller than lead time	Use Eq. 6.6 to compute standard deviation of lead time demand
If demand period is greater than lead time	Use Eq. 6.7 to compute standard deviation of lead time demand

Case 2: Demand period is greater than lead time

Consider a problem where the demand period is greater than lead time (e.g. demand is annual and lead time is specified in days). If the standard deviation of demand period is given and we need to determine the standard deviation of lead time demand, we can use the following:

$$\sigma_{\mathrm{d}} = \frac{\sigma_{\mathrm{i}}}{\sqrt{n}} \tag{6.7}$$

where n is the number of lead time periods that make up the demand period. For example, if demand period is in months and lead time is in weeks, then $n = 4$ (i.e., 4 weeks make up a month). General rules to be followed are shown in Table 6.8

Solved Problem 6.3

The daily demand for an item is normally distributed with a mean of 100 units and a standard deviation of 3 units. If the procurement lead time is 6 days, compute the standard deviation of lead time demand.

Solution

In this problem, demand period is 1 day (daily) and lead time is 6 days. Since demand period is smaller than lead time, we use Eq. 6.6 to obtain the standard deviation of lead time demand. The standard deviation of the daily demand is 3. The standard deviation of lead time demand, assuming demand for each day is independent, is

$$\sigma_{\mathrm{d}} = \sqrt{3^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2} = \sqrt{54} = 7.35 \text{ units}$$

The standard deviation of lead time demand is 7.35 units.

Solved Problem 6.4

The demand for an item in a month is normally distributed with a mean of 100 units and a standard deviation of 3 units. If the lead time is 1 week, compute the standard deviation of the lead time demand.

Solution

In this problem, the demand period is 1 month and lead time is 1 week. Since demand period is greater than the lead time, we use Eq. 6.7 to obtain the standard deviation of lead time demand. We assume there are 4 weeks in a month, so $n = 4$. Using Eq. 6.7, and substituting the values, we get

$$\sigma_1 = \frac{3}{\sqrt{4}} = 1.5 \text{ units}$$

The standard deviation of lead time demand is 1.5 units.

Solved Problem 6.5

Daily demand for a certain item is governed by a normal distribution with a mean of 100 units and a standard deviation of 8 units. The time between placing an order and receipt of supply is fairly constant at 5 days. The ordering cost per order is $25 and the cost per item is $50. Assume 360 working days in a year and 20% interest rate per year. Answer the following questions (no backlogging allowed):

(a) Compute the optimal order quantity.
(b) Compute the reorder level to satisfy an 85% probability of not stocking out during the lead time.
(c) Compute the fill rate for the replenishment policy.
(d) What safety stock would need to be maintained to achieve a fill rate of 0.99?

Solution

The following data are available with us:

- Mean daily demand, $\bar{d}$, is 100 units (Annual demand is 36,000 units per year).
- Standard deviation of daily demand, σ_i, is 8 units.
- Lead time is 5 days.
- Ordering cost, C_o, is $25 per order.
- Cost of item, C, is $50.
- Carrying interest rate, i, is 0.20.

(a) Economic Order Quantity

Using Eq. 3.8 (Chap. 3), we can determine the optimal order quantity. Substituting the values, we get,

$$Q = \sqrt{\frac{2 \times 36000 \times 25}{0.20 \times 50}} = 424 \text{ units}$$

(b) Reorder level

In this problem, the demand period is 1 day while the lead time is 5 days. Since demand period is smaller than the lead time, we use Eq. 6.6 to compute the standard deviation of lead time demand:

$$\sigma_d = \sqrt{\sum_{i=1}^{5} \sigma_i^2} = \sqrt{8^2 + 8^2 + 8^2 + 8^2 + 8^2} = 17.9$$

The standard deviation of the lead time demand, σ_d, is 17.9 units.

The problem states that an 85% probability of not stocking out during lead time is desired. We can use the NORM.S.INV(p) function in MS Excel to obtain the standard normal variate for a specified probability (0.85). We get

$$z = \text{NORM.S.INV}(0.85) = 1.04$$

We can now use Eq. 6.4 to determine the reorder level, s. Substituting the values in Eq. 6.4, we get

$$s = \bar{d}L + z\sigma_{\mathrm{d}} = (100 \times 5) + (1.04 \times 17.9) = 519 \text{ units}$$

The ordering policy for the given problem is as follows:

Place an order for 424 units when the inventory level reaches 519 units.

Alternatively, we can use Eq. 6.5 to obtain the reorder level when the standard deviation of daily demand is specified. Note that we have the following data:

- Mean daily demand, $\bar{d}$, is 100 units
- Standard deviation of daily demand is 8 units
- Lead time is 5 days

Substituting the values in Eq. 6.5, we get

$$s = (100 \times 5) + \left(1.04 \times 8 \times \sqrt{5}\right) = 519 \text{ units}$$

Note that this is the same answer we got using Eq. 6.4.

(c) Fill rate

To determine the fill rate, we have to first estimate the expected shortage per order cycle, E_{s}. We can use the formula given in the box earlier in this chapter. Substituting the values, we get

$$E_{\mathrm{s}} = 17.9 \times \mathrm{NORM.DIST}\left(\frac{19}{17.9}, 0, 1, 0\right) - 19\left[1 - \mathrm{NORM.DIST}\left(\frac{19}{17.9}, 0, 1, 1\right)\right]$$

or $E_{\mathrm{s}} = 1.32$. The fill rate can be determined using the following formula:

$$f_{\mathrm{r}} = \frac{Q - E_{\mathrm{s}}}{Q} = \frac{424 - 1.32}{424} = 0.9968$$

or the fill rate is 99.68%.

(d) Safety stock to achieve a fill rate of 99%

To achieve the desired fill rate of 0.99, we need to first compute the expected shortages. We know that

$$f_{\mathrm{r}} = \frac{Q - E_{\mathrm{s}}}{Q}$$

Substituting the values in the above equation we get

$$0.99 = \frac{424 - E_s}{424}$$

or $E_s = 3$. We also know that

$$E_s = \sigma \times \mathrm{NORMDIST}\left(\frac{\mathrm{SS}}{\sigma}, 0, 1, 0\right) - \mathrm{SS}\left[1 - \mathrm{NORMDIST}\left(\frac{\mathrm{SS}}{\sigma}, 0, 1, 1\right)\right]$$

Substituting the available values, we get

$$3 = 17.9 \times \mathrm{NORM.DIST}\left(\frac{\mathrm{SS}}{17.9}, 0, 1, 0\right) - \mathrm{SS}\left[1 - \mathrm{NORM.DIST}\left(\frac{\mathrm{SS}}{17.9}, 0, 1, 1\right)\right]$$

We can use the GOAL SEEK function[4] in MS Excel to obtain the value of SS. In this case, we obtain a value of SS = 10.8 (or 11 units) to achieve a fill rate of 99%.

Solved Problem 6.6

The daily demand for an item is 20 units. The procurement lead time for the item is 10 days. The standard deviation of the lead time demand is 12 units. Determine the order level for this situation to satisfy an 85% probability of not stocking out during lead time.

Solution

The reorder level, s, can be computed using Eq. 6.4. In this problem, the following data are given:

- Daily demand $\bar{d} = 20$ units.
- Lead time $L = 10$ days.
- The standard deviation of demand during lead time $\sigma_d$is directly specified (12 units).
- $z = \mathrm{NORM.S.INV}(0.85) = 1.04$

Substituting the above values in Eq. 6.4, we get

$$s = (20 \times 10) + (1.04 \times 12) = 213 \text{ units}$$

The reorder level is 213 units. The safety stock is 13 units.

Solved Problem 6.7

The daily demand for an item is 20 units. The procurement lead time for the item is 10 days and the standard deviation of the demand during the lead time is 12 units. Determine the cycle service level if the reorder level is 213 units.

[4]See Appendix 6B.

Solution
We have the following data with us:

- Daily demand $\bar{d} = 20$ units.
- Lead time $L = 10$ days.
- The standard deviation of demand during lead time $\sigma_d$is directly specified (12 units).
- Reorder level is 213 units.

The demand during lead time is $20 \times 10 = 200$ units. We can use MS Excel function NORM.DIST to find the value of the cycle service level. The syntax is as follows:

$$\text{Cycle Service Level} = \text{NORM.DIST}\left(\text{ROP}, \bar{d}L, \sigma_\text{d}, \text{FLAG}\right)$$

Using the MS Excel function with the values obtained, we get

$$= \text{NORM.DIST}(213, 200, 12, \text{TRUE}) = 0.86$$

The cycle service level is 86%.[5]

Solved Problem 6.8
The daily demand for an item is 20 units. The procurement lead time for the item is 10 days. The standard deviation of the lead time demand is 12 units. Determine the fill rate if an order for 300 units is placed each time the inventory falls to 213 units.

Solution
We have the following data with us:

- Order size $Q = 300$ units.
- Daily demand $\bar{d} = 20$ units.
- Lead time $L = 10$ days.
- The standard deviation of demand during lead time $\sigma_d$is directly specified (12 units).
- Reorder level is 213 units.

The safety stock is $213 - 200 = 13$ units. We can use the MS Excel function illustrated in the box (earlier in this chapter) to compute the fill rate:

$$E_\text{s} = 12 \times \text{NORM.DIST}\left(\frac{13}{12}, 0, 1, 0\right) - 12\left[1 - \text{NORM.DIST}\left(\frac{13}{12}, 0, 1, 1\right)\right]$$

or $E_\text{s} = 0.85$. Fill rate is given by

[5]For the same replenishment policy, the cycle service level was 85% in Solved Problem 6.6. This is due to the rounding-off of the reorder level.

$$\text{Fr} = \frac{Q - E_s}{Q} = \frac{300 - 0.85}{300} = 0.997$$

The fill rate is 99.7%.[6]

6.4.2.2 Reorder Level – Variable Lead Time, Constant Demand

In this section, we consider the case where the replenishment lead time is variable and the demand is constant. Following assumptions have been made is determining the reorder level for this case:

- The inventory system is reviewed continuously, i.e., the inventory level is known at all times.
- The inventory system involves a single item.
- Demand for the item is known and constant.
- Lead time is random but the distribution governing the lead time demand is known (normally distributed).

The order size can be determined using Eq. 3.8 as discussed in Chap. 3. When the demand is constant and lead time is variable, the reorder level, s, is given by

$$s = d\bar{L} + zd\sigma_L \tag{6.8}$$

where

- d is the daily demand;
- $\bar{L}$ is the average lead time;
- z is the standardized normal variate;
- The term σ_L represents the standard deviation of lead time.

In Eq. 6.8, the term $zd\sigma_L$ represents the safety stock.

Solved Problem 6.9
Hospital Angeles performs 10 heart surgeries each day with one stent for each surgery. The hospital procures stents from Germany. The procurement lead time is normally distributed with a mean of 10 days and a standard deviation of 3 days. If the hospital management wants a 95% probability of not stocking out during lead time, compute the safety stock and the reorder level for stents.

Solution
In this problem,

- d is 10 stents (constant).
- $\bar{L}$ is 10 days.

[6] Compare this with the cycle service level in Solved Problem 6.7 which was 86% while the fill rate is 99.7%.

- z, computed using NORM.S.INV(0.95) function, is 1.64.
- σ_L is 3 days.

Substituting the above in Eq. 6.8, we get

$$s = (10 \times 10) + (1.64 \times 10 \times 3) = 149 \text{ stents}$$

The safety stock is 49 units and the reorder level is 149 stents.

6.4.2.3 Reorder Level – Variable Demand and Lead Time

In the previous sections, we considered the variability of either demand or the lead time. In this section, we consider the case where both demand and lead times are variable. Assuming that the demand and lead times are normally distributed, the reorder level, s, is given by

$$s = \bar{d}\bar{L} + z\sqrt{(\sigma_d^2)\bar{L} + (\sigma_L^2)\bar{d}^2} \tag{6.9}$$

where

- $\bar{d}$ is the average demand per period;
- σ_d is the standard deviation of demand per period;
- $\bar{L}$ is the average replenishment lead time;
- σ_L is the standard deviation of lead time.

The term $\sqrt{(\sigma_d^2)\bar{L} + (\sigma_L^2)\bar{d}^2}$ is the standard deviation of demand during lead time and the term $z\sqrt{(\sigma_d^2)\bar{L} + (\sigma_L^2)\bar{d}^2}$ represents the safety stock.

Solved Problem 6.10

The daily demand experienced by small home computer assembler is normally distributed with a mean of 20 units and a standard deviation of 6 units. The assembler sources the RAM for the computer from a supplier in the local market. The lead time for supply of the RAM chips is also normally distributed with a mean of 3 days and a standard deviation of 1 day. If the assembler desires to ensure a 90% probability of not stocking out during lead time, compute the following:

- Reorder level, s
- Safety stock, SS

Solution

Since both the demand and lead time are variable, we use Eq. 6.9 to compute the reorder level. The value of z for a probability of 0.90 is

$$z = \text{NORM.S.INV}(0.90) = 1.28$$

Substituting the values, we get

$$s = (20 \times 3) + 1.28\sqrt{\left(6^2 \times 3\right) + \left(1^2 \times 20^2\right)} = 60 + 28.8 = 88.8 \text{ units}$$

The reorder level is 89 units while the safety stock is 29 units to maintain a desired cycle service level of 90%.

Solved Problem 6.11

Compute the reorder level and safety stock using the following data, assuming demand and lead times are normally distributed:

- Average demand is 30 units per day
- Standard deviation of demand is 5 per day
- Average lead time is 10 days
- Standard deviation of lead time is 3 days
- 85% probability of not stocking out during lead time is desired

Solution

Since both the demand and lead time are variable, we use Eq. 6.9 to compute the reorder level. The value of z for a probability of 0.85 is follows:

$$z = \text{NORM.S.INV}(0.85) = 1.03$$

Substituting the given values in Eq. 6.9, we get

$$s = (30 \times 10) + 1.03\sqrt{\left(5^2 \times 10\right) + \left(3^2 \times 30^2\right)} = 300 + 95 = 395 \text{ units}$$

The reorder level is 395 units, while the safety stock is 95 units.

6.5 Reorder Level – Planned Shortages Allowed

When planned shortages, or backlogging, is allowed we can use the *EOQ* model with planned shortages (discussed in Chap. 3) to compute the order size. The formula to compute the order size when planned shortages are allowed is as follows:

$$Q = \sqrt{\frac{2DC_o(C_h + C_s)}{C_h C_s}}$$

We will continue to use Eq. 6.4 (or Eq. 6.5) to determine the order level when the demand is variable.

Solved Problem 6.12

Steren sells just one type of electric adaptors that are used in households and offices. Demand for adaptors is normally distributed with a mean of 500 units per month and a standard deviation of 20 units per month. The inventory carrying cost, C_{h}, is \$0.5 per piece per month, and ordering cost per order, C_{o}, is \$350 per order. If backlogging is allowed and the shortage cost is \$5 per adaptor per month, compute the *EOQ* and the reorder level for an 85% probability of not stocking out during the lead time. The lead time is 1 month.

Solution

(a) Economic Order Quantity

Substituting the given values in Eq. 3.20, we get

$$Q = \sqrt{\frac{2DC_{\mathrm{o}}(C_{\mathrm{h}} + C_{\mathrm{s}})}{C_{\mathrm{h}}C_{\mathrm{s}}}} = \sqrt{\frac{2 \times 500 \times 350 \times (0.5 + 5)}{0.5 \times 5}} = 877 \text{ units}$$

(b) Reorder Level

Reorder level can be computed using Eq. 6.4. z = 1.04 for a probability of 0.85. Substituting the values in Eq. 6.4, we get

$$s = \bar{d}L + z\sigma_{\mathrm{d}} = \left(\frac{500}{30} \times 30\right) + (1.04 \times 20) = 521 \text{ units}$$

6.6 Periodic Review-based Models

Recall that in periodic review models, the inventory levels are reviewed at predetermined, discrete times such as every Friday, the last working day of every month, etc. A review of on-hand inventory is conducted at time, T, and an order is placed. The order quantity is equivalent to the quantity that is needed to bring the inventory level back up to a prespecified maximum level, S. Therefore, in periodic review models, the order quantities generated varies each period. In this section, we describe methods for computing the order quantity, safety stock, and/or the maximum inventory for the following scenarios (Sharma 2006, Venkataraman and Pinto 2017):

- Demand is variable, lead time is constant.
- Lead time is variable, demand is constant.
- Both demand and lead times are variable.

6.6.1 Order Quantity – Variable Demand and Constant Lead Time

Let us consider the case when demand is variable and the lead time is constant. We make the following assumptions:

- The inventory system is reviewed periodically with a review period of T days.
- The inventory system involves a single item.
- Demand for the item is random, but the distribution governing the demand is known or can be estimated. We assume the demand is normally distributed with a known mean and standard deviation.
- Lead time, L, is known and constant.
- An order of size Q is placed on completion of the review. The size of the order is such that it brings the inventory on hand, I, up to a maximum level, S, specified by the management. Q will, therefore, vary from period to period.

The first step in solving the variable demand, constant lead time review period problem is to determine the optimal review period T. This can be computed using Eq. 3.24 from Chap. 3, which is reproduced as follows:

$$T = \sqrt{\frac{2C_\mathrm{o}}{iDC}}$$

The order quantity Q is given by:

$$Q = \bar{d}(T + L) + z\sigma_\mathrm{d}\sqrt{(T + L)} - I \tag{6.10}$$

where

- z is the standard normal variate for a specified probability;
- σ_d is the standard deviation of the demand; and
- I is the on-hand inventory including inventory on order.

Safety stock (SS) and maximum inventory level (S) can be determined using the following equations:

$$SS = z\sigma_\mathrm{d}\sqrt{(T + L)} \tag{6.10a}$$

$$S = \bar{d}(T + L) + z\sigma_\mathrm{d}\sqrt{(T + L)} \tag{6.10b}$$

Solved Problem 6.13

A retail shop uses a periodic review inventory system. The annual demand for a consumer product sold by the shop is 5500 units, the ordering cost is \$43 per order, and the inventory carrying rate is 35% per year. The product costs \$100 per unit. The daily demand for the product is normally distributed with a mean of 15 units and a standard deviation of 3 units. If the procurement lead time is a constant 5 days and the retail shop desires a 90% probability of no stockout during lead time, compute:

(a) Optimal review period
(b) Safety stock, SS
(c) Maximum inventory, S
(d) Order quantity, if the on-hand inventory is 24 units.
(e) New safety stock if the shop intends to use a new service level criterion – that is to limit stockout to 1 order cycle per year (instead of 90% service level)

Solution

(a) Optimal Review Period

Using Eq. 3.24, we can compute the optimal review period, which is

$$T = \sqrt{\frac{2 \times 43}{0.35 \times 5500 \times 100}} = 0.0211 \text{ years}$$

The optimal review period is 0.0211 × 365 days=7.7 days.

(b) Safety Stock, SS

Safety stock can be determined using Eq. 6.10a. The value of $z = \text{NORM.S.INV}(0.90) = 1.28$. The lead time is a constant 5 days and the standard deviation of demand is 3 units. Substituting the values, we get

$$\text{SS} = 1.28 \times 3 \times \sqrt{(7.7 + 5)} = 13.6 \text{ units}$$

(c) Maximum Inventory, S

Maximum inventory, S, can be determined using Eq. 6.10b. The average daily demand is 15 units. Therefore, we have

$$S = 15 \times (7.7 + 5) + 13.6 = 204 \text{ units}$$

(d) Order Quantity

The order quantity can be determined using Eq. 6.10. Since the on-hand inventory is 24 units, we have

$$Q = 204 - 24 = 180 \text{ units}$$

The quantity to be ordered at the review point (24 units on hand) is 180 units.

(e) Safety stock for new service-level criterion

Since Q is 204 units, the number of orders to be placed is

$$\frac{5500}{204} = 26.9 \text{ orders per year}$$

If one stockout is allowed per year, then the service level expected is

$$= \frac{25.9}{26.9} = 0.96$$

The standard normal variate can be found using the MS Excel function, NORM . S . INV(0.96) = 1.75. Substituting the values, we get

$$\text{SS} = 1.75 \times 3 \times \sqrt{(7.7 + 5)} = 18.6 \text{ units}$$

Solved Problem 6.14

Weekly demand for a certain item at a firm follows a normal distribution with a mean of 50 units and a standard deviation of 5 units. The optimal review period is 3 weeks while the lead time is constant at 2 weeks. The firm wants to ensure there is no stockout 95% of the time during the lead time. Compute the safety stock, target inventory, and the order quantity if at the time of review there are 58 units in store.

Solution

(a) Safety Stock, SS

Safety stock can be determined using Eq. 6.10a. The desired service level is 95%, and the value of z = NORM . S . INV(0.95) = 1.28. The lead time is a constant 2 weeks, the review period is 3 weeks, and the standard deviation of demand is 5 units. Substituting the values, we get

$$\text{SS} = 1.64 \times 5 \times \sqrt{(3 + 2)} = 18.3 \text{ units}$$

(b) Maximum Inventory Level, S

Maximum inventory, S, can be determined using Eq. 6.10b. The average weekly demand is 50 units. Therefore, we have

$$S = 50 \times (3 + 2) + 18.3 = 268 \text{ units}$$

(c) Order Quantity

The order quantity can be determined using Eq. 6.10. Since the on-hand inventory is 58 units, we have

$$Q = 268 - 58 = 210 \text{ units}$$

The quantity to be ordered at the review point when 58 units are available is 210 units.

6.6.2 Order Quantity – Constant Demand and Variable Lead Time

Let us consider the case when demand is constant and the lead time is variable. We make the following assumptions:

- The inventory system is reviewed periodically with a review period of T days.
- The inventory system involves a single item.
- Demand for the item is constant.
- Lead time, L, is variable, but the distribution governing the lead time is known or can be estimated. We assume the lead time is normally distributed with a known mean and standard deviation.
- An order of size Q is placed on completion of review. The size of the order is such that it brings the inventory on hand, I, up to a target level, TI, specified by the management. Q will, therefore, vary from period to period.

  The order quantity Q in this case can be given by

$$Q = d(T + L) + zd\sigma_{\mathrm{L}} - I \tag{6.11}$$

where

- z is the standard normal variate for a desired service level;
- σ_{L} is the standard deviation of the lead time;
- I is the on-hand inventory including inventory on order.

In Eq. 6.11, the term $zd\sigma_{\mathrm{L}}$ is the safety stock, SS, while the term $d(T+L)+zd\sigma_{\mathrm{L}}$ is the maximum inventory level, S.

$$\mathrm{SS} = zd\sigma_L \tag{6.11a}$$

$$S = d(T + L) + zd\sigma_{\mathrm{L}} \tag{6.11b}$$

Solved Problem 6.15

The annual demand for a consumer product sold by a retailer is 5500 units, the ordering cost is \$43 per order, and the inventory carrying rate is 35% per year. The product costs \$100 per unit. The daily demand for the product is constant at 15 units. If the procurement lead time is normally distributed with a mean of 5 days and a standard deviation of 1 day, compute the following if the retailer wishes a 90% service level:

(a) Optimal Review Period
(b) Safety Stock
(c) Maximum Inventory
(d) Order Quantity, if the on-hand inventory is 24 units

Solution

(a) Optimal Review Period

Using Eq. 3.24, we can compute the optimal review period, which is

$$T = \sqrt{\frac{2 \times 43}{0.35 \times 5500 \times 100}} = 0.0211 \text{ years}$$

The optimal review period is 0.0211 × 365 days=7.7 days.

(b) Safety Stock, SS

We can determine safety stock using Eq. 6.11a. For the desired service level of 90%, the value of $z = \text{NORM.S.INV}(0.90) = 1.28$. The lead time standard deviation is 1 day. Substituting the values, we get

$$\text{SS} = 1.28 \times 15 \times 1 = 19.2 \text{ units}$$

(c) Maximum Inventory, S

We can determine maximum inventory using Eq. 6.11b. The daily demand is 15 units. Therefore, we have

$$S = 15 \times (7.7 + 5) + 19.2 = 210 \text{ units}$$

(d) Order Quantity

The order quantity can be determined using Eq. 6.11. Since the on-hand inventory is 24 units, we have

$$Q = 210 - 24 = 186 \text{ units}$$

The quantity to be ordered at the review point of 24 units on hand is 186 units.

6.6.3 *Order Quantity – Variable Demand and Lead Time*

Let us consider the case when both demand and lead times are variable. We make the following assumptions:

- The inventory system is reviewed periodically with a review period of T days.
- The inventory system involves a single item.
- Demand for the item is variable but the distribution governing the demand is known or can be estimated. We assume the demand is normally distributed with a known mean and standard deviation.
- Lead time, L, is variable but the distribution governing the lead time is known or can be estimated. We assume the lead time is normally distributed with a known mean and standard deviation.

- An order of size Q is placed on completion of review. The size of the order is such that it brings the inventory on hand, I, up to a maximum level, S, specified by the management. Q will, therefore, vary from period to period.

  In this scenario, the order quantity Q can be given by

$$Q = \bar{d}(T + L) + z\sqrt{\left[(T + L)\sigma_{\mathrm{d}}^2\right] + \bar{d}^2\sigma_{\mathrm{L}}^2} - I \tag{6.12}$$

where

- z is the standard normal variate;
- σ_{d} and σ_{L} are the standard deviations of the demand and lead time, respectively;
- I is the on-hand inventory including inventory on order.

  Safety stock and maximum inventory level for this condition are as follows:

$$\mathrm{SS} = \sqrt{\left[(T + L)\sigma_{\mathrm{d}}^2\right] + \bar{d}^2\sigma_{\mathrm{L}}^2} \tag{6.12a}$$

$$S = \bar{d}(T + L) + z\sqrt{\left[(T + L)\sigma_{\mathrm{d}}^2\right] + \bar{d}^2\sigma_{\mathrm{L}}^2} \tag{6.12b}$$

Solved Problem 6.16

A retail shop uses a periodic review inventory system. The annual demand for a consumer product sold by the shop is 5500 units, the ordering cost is \$43 per order, and the inventory carrying rate is 35% per year. The product costs \$100 per unit. The daily demand for the product is normally distributed with a mean of 15 units and a standard deviation of 3 units. If the procurement lead time is normally distributed with a mean of 5 days and a standard deviation of 1 day, compute the safety stock if the retailer wishes a 90% probability of not stocking out during lead time.

Solution

Eq. 6.12a can be used to determine safety stock.

We have already computed the optimal review period (7.7 days).

σ_{d} is 3 and σ_{L} is 1.

Mean demand is 15 and mean lead time is 5.

For the desired service level 90%, $z = \mathrm{NORM.S.INV}(0.90) = 1.28$. The lead time standard deviation is 1 day. Substituting the values, we get

$$SS = 1.28\sqrt{\left[(7.7 + 5) \times 3^2\right] + 15^2 \times 1^2} = 1.28 \times 18.4 = 24 \text{ units}$$

The safety stock is 24 units.

6.7 Summary

In this chapter, we discussed inventory models that account for the uncertainty of demand (and lead time). When demand for an item is uncertain, there could be situations when demand could exceed expectations or orders may arrive late. This situation results in a stockout that may, in turn, result in business losses. To reduce the impact of uncertainty on demand, inventory managers carry additional inventory – called the safety stock. In this chapter,we discussed how we could use the historical data and fit it into a probability distribution (discrete as well as continuous) to determine the size of safety stock.

The following three cases were examined in detail, for continuous review as well as periodic review-based systems:

- Demand is uncertain and lead time is constant
- Demand is constant and lead time is uncertain
- Both demand and lead time are uncertain

6.8 Case Study – Trading

Juan Manuel is the proprietor of a 6-year-old trading firm based in Leon, Mexico. Juan buys fractional horsepower (FHP) motors from the cheaper local market and sells them to manufacturers of electrical switch gears at a higher rate globally. This model has worked very well for Juan. Since he uses a local, reliable supplier, the replenishment lead time is a constant 1 month. Juan has not been using any scientific inventory management techniques. The size of the order is arbitrary, and neither is the cycle time fixed. This was the way he worked in the initial years after starting the trading business.

Six years hence, Juan's business has grown. Several global brands now source their requirements of FHP motors from him, and they are not very tolerant to late deliveries. Being in the business for the last 6 years has made Juan realize the importance of scientific inventory management. He knows that a shortage of inventory, when demand occurs, would be catastrophic. While he understands meeting every customer demand would be impossible, he is keen, however, to minimize such undesirable events.

Juan has, therefore, set up on a journey to improve the way he manages inventory at his firm. He starts looking at his invoices, customer orders, and everything he has got on his computer. After a week, Juan has been able to tabulate the sales data for the last 36 months (3 years) for one of the fast-moving models – FHPX3. This is shown in the folowing table:

173	215	260	305	273	345	221	265	311	296	183	338
385	285	355	204	376	179	404	229	289	151	325	440
345	459	195	375	232	398	299	419	340	425	245	261

Case study questions
Draw a cumulative probability plot and determine the following:

(a) The average inventory of FHPX3 motor.
(b) What safety stock would Juan need to maintain if he desires a cycle service level of 90%?

Juan incurs an administrative ordering cost of $150 per order. Each motor costs $1200 and Juan uses an inventory carrying rate of 25% per year for accounting purposes. Juan plans to use a continuous review system. Compute the following assuming the monthly demand follows a normal distribution with a mean of 300 units and a standard deviation of 80 units:

(c) Reorder level, if the procurement lead time is 1 month and a desire of 85% probability of not stocking out during lead time.

6.9 Practice Problems

Problem 6.1
The annual demand for a certain item is 1200 units. The procurement lead time is 10 days. If the standard deviation of lead time demand is 8 units and a service level of 0.95 is desired by the management, determine the reorder level. Assume 300 workdays in a year and demand to be normally distributed.

Answer
Reorder level is 313 units.

Problem 6.2
The daily demand for a certain item is normally distributed with a mean of 50 units and standard deviation of 3. The procurement lead time is constant 5 days. If a service level of 0.95 is desired by the management, determine the reorder level.

Answer

$$s = \bar{d}L + z\sigma_{\mathrm{i}}\sqrt{L} = 50 \times 5 + \left(1.65 \times 3\sqrt{5}\right) = 261 \text{ units}$$

Problem 6.3
The demand for an item at a retail shop is a constant 30 per day. The procurement lead time is normally distributed with a mean of 10 days and a standard deviation of 2 days. If the retailer wants a service level of 95%, compute the safety stock that the retailer would need to maintain. Also, compute the reorder level.

Answer

$$s = d\bar{L} + zd\sigma_L = 30 \times 10 + (1.65 \times 30 \times 2) = 399 \text{ units}$$
$$\text{SS} = (1.65 \times 30 \times 2) = 99 \text{ units}$$

Problem 6.4

Demand for an item is 200 per day and standard deviation is 60 per day. Lead time is 15 days constant. If s is 3200 units, what cycle service level is implied?

Answer

- Note that the standard deviation of demand during lead time is not specified. This needs to be calculated.
- Use $NORM.DIST\left(3200, 200 \times 15, 60\sqrt{15}, TRUE\right)$. The implied cycle service level is 80.56%.

Problem 6.5

The daily demand for an item is normally distributed with a mean of 30 units and a standard deviation of 4 units. The procurement lead time for an item is also normally distributed with a mean of 5 days and a standard deviation of 1 day. Compute the reorder level if a 90% probability of not stocking out during lead time is desired.

Answer

$$s = (30 \times 5) + 1.28\sqrt{\left(4^2 \times 5\right) + \left(1^2 \times 30^2\right)} = 190 \ \text{units}$$

Problem 6.6

Weekly demand for a certain item at a firm follows a normal distribution with a mean of 200 units and a standard deviation of 50 units. The optimal review period is 4 weeks while the lead time is constant at 3 weeks. If the firm wants to ensure a cycle service level of 98%, compute the safety stock, maximum inventory, and the order quantity, if at the time of review there are 100 units in store.

Answer

- Safety stock is $z\sigma_{\text{d}}\sqrt{(T+L)} = 2.05 \times 50 \times \sqrt{4+3} = 271$ units.
- Maximum inventory is $\bar{d}\,(T+L) + z\sigma_{\text{d}}\sqrt{(T+L)} = 200 \times 7 + 271 = 1671$ units.
- Order quantity is $\bar{d}\,(T+L) + z\sigma_{\text{d}}\sqrt{(T+L)} - I = 1671 - 100 = 1571$ units.

Appendix 6A: EOQ – When Shortage Costs Are Known

Monks (1987) suggests that when the stockout costs are known accurately, they can be included in the *EOQ* formula just like the ordering cost. We start by computing the *EOQ*, which we know can be computed using Eq. 3.8, reproduced as follows:

$$Q = \sqrt{\frac{2DC_o}{iC}}$$

Next, we compute the implied stockout cost per order, which is given by

$$C_s = \frac{iCQ^2}{2D} \tag{6.13}$$

where C_s is the stockout cost per order. We can use the implied stockout cost in the following equation to obtain the revised *EOQ* with stockout costs:

$$Q = \sqrt{\frac{2D(C_o + C_s)}{iC}} \tag{6.14}$$

It should be noted that this is only an approximate solution and will require one or more iterations. The following solved problem illustrates the application of this equation.

Solved Problem 6.17
A firm has an annual demand of 1200 units that costs \$65 and an ordering cost of \$20 per order. If the firm uses an inventory rate of 25% per year, compute the implied stockout cost and the order quantity.

Solution
As described earlier, this problem will need to be solved iteratively. We start by computing the *EOQ without* shortages. We have the following information with us:

- D is 1200 units
- C_o is \$20
- i, the inventory rate, is 25% per year
- C, the item cost, is \$65

Using Eq. 3.8, we can compute *EOQ* without shortages. Substituting the values, we get

$$Q = \sqrt{\frac{2 \times 1200 \times 20}{0.25 \times 65}} = 54 \text{ units}$$

The implied stockout cost per order can be determined using Eq. 6.13. Substituting the values, we get

$$= \frac{0.25 \times 65 \times (54)^2}{2 \times 1200} = \$19.75 \text{ per order}$$

Finally, we compute the revised EOQ with shortages using Eq. 6.14. Substituting the values, we get

$$Q = \sqrt{\frac{2D(C_o + C_s)}{iC}} = \sqrt{\frac{2 \times 1200 \times (20 + 19.75)}{0.25 \times 65}} = 77 \text{ units}$$

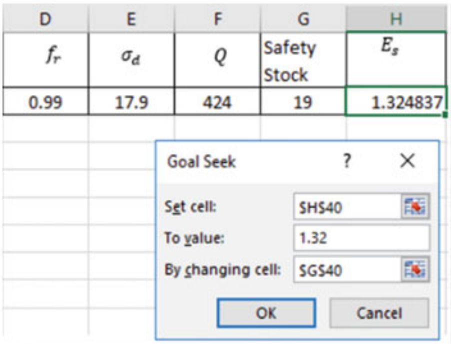
The order quantity when the cost of stockout is known is 77 units.

Appendix 6B: Using GOAL SEEK function in MS Excel

The procedure to compute safety stock expected to satisfy the desired *fill rate* is complex. However, we may use the NORM.DIST and GOAL SEEK functions in MS Excel to solve this problem. When the demand for an item is normally distributed with a demand D, standard deviation, and safety stock SS, the expected shortage per order cycle can be given by (See Chopra and Meindel 2010):

$$E_s = -1 \times SS \left[1 - \text{NORMDIST}\left(\frac{\text{SS}}{\sigma}, 0, 1, 1 \right) + \sigma \times \text{NORMDIST}\left(\frac{\text{SS}}{\sigma}, 0, 1, 0 \right) \right]$$

The GOAL SEEK function (navigate to *Data* tab, and then to *What-if Analysis* (in the forecast group)) may be used to determine the optimal value for SS.



The formula in cell G40 is

- G40*(1-NORM.DIST(G40/E40,0,1,TRUE)) + E40*NORM.DIST(G40/E40,0,1,0)

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Part III
Multi-item Inventory Models

Chapter 7
Multi-item Inventory Models Subject to Constraints

7.1 Introduction

Inventory managers at Rosetta's manage a large number of items in stock such as corn flour, vegetable oil, cooking gas, yeast, and several others. Stocking decisions of these items are dependent on each other. For example, the decision to stock a certain amount of vegetable oil depends on the amount of corn flour being stocked. What this means for Rosetta's is that stocking large quantities of vegetable oil and small amount of corn flour is not really going to help. Another problem being faced by them is the fact that the management wants to restrict the amount of money they lock in inventory. In these situations, what should be their replenishment policy?
In this chapter we address multi-item inventory decisions that are subject to one or more resource constraints.

Inventory managers are expected to manage a large number of items. Stocking decisions are usually made jointly among the many items managed at a given location (Muckstadt and Sapra 2010). Consider the situation at Rosetta's (see box above) where they decide to order vegetable oil and corn flour, both based on the EOQ. Table 7.1 lists the inventory data that are needed for computation of order size.

Substituting the above values in Eq. 3.8 (from Chap. 3), we can determine the EOQs for both these items, which are shown in Table 7.2.

Using the EOQs, we can compute the amount Rosetta's would need to invest in the inventory of these two items, which is

$$= \frac{1}{2}[(438 \times 20) + (666 \times 30)] = \$14,370$$

D. Shenoy, R. Rosas, *Problems & Solutions in Inventory Management*,
DOI 10.1007/978-3-319-65696-0_7

Table 7.1 Inventory data – Interacting inventory items

Parameter	Vegetable oil	Corn flour
Annual demand	7200 liters	25,000 kg
Ordering cost ($ per order)	$80	$80
Carrying rate (per year)	30%	30%
Unit price	20	30

Table 7.2 EOQ and inventory investment

Parameter	Vegetable oil	Corn flour
EOQ	438 liters	666 kg
Average investment	$4380	$9990

If Rosetta's decide to use EOQ to manage their inventory system, they would lock up $14,370 in the inventory of these two items, on an average. However, the management of Rosetta's may want to limit the investment to a smaller amount, say $12,000. Because of the interdependency between vegetable oil and corn flour, Rosetta's would need to adjust the order size downward for both the items to satisfy the constraint of $12,000.

What we discussed above was a case of budgetary constraint. Manufacturing organizations work under several constraints, including budgetary constraint. In this chapter, we will review multi-item inventory models that are subject to the following constraints:

- Budget constraint, or the average investment in inventory
- Space constraint, or the amount of space available to stock the items
- Number of orders constraint, or the number of orders that may be placed each year

To start with, we will review models with each of the constraints taken individually. Later in this chapter, we will discuss models where inventory decisions are subject to more than one of these constraints.

7.2 Budget Constraint

Consider an inventory system with n items. Let B be the maximum amount of money that can be invested in stock. If Q_j is the optimal order size, C_{oj} is the ordering cost, D_j is the annual demand, i is the inventory carrying rate per year, and C_j is the unit cost for the jth item in the system, the problem can be mathematically formulated as follows:

$$\text{Min} \sum_{j=1}^{n} \left(\frac{D_j}{Q_j} C_{oj} + i \frac{Q_j}{2} C_j \right)$$

subject to

$$\sum_{j=1}^{n} \frac{Q_j}{2} C_j \leq B$$

The objective function measures the total inventory costs while the term in the constraint $\sum_{j=1}^{n} \frac{Q_j}{2} C_j$ restricts the investment in inventory to B. If the constraint is satisfied by individual optimal order quantities, we not only have the individual optimal Q values but also those that are feasible Q values. However, when the constraint is not satisfied, the individual Q values obtained are infeasible, and we rewrite the constraint by replacing the inequality as

$$\sum_{j=1}^{n} \frac{Q_j}{2} C_j = B$$

To solve this problem, we need to introduce a Lagrangean multiplier[1], θ. The objective now is to find the optimal as well as feasible values for each of j. The Lagrangean function can be written as

$$L = \sum_{i=j}^{n} \frac{D_j}{Q_j} C_{oj} + \sum_{i=j}^{n} i\frac{Q}{2} C_j + \theta \sum_{i=j}^{n} \frac{Q_j}{2} C_j - B \tag{7.1}$$

The optimal values for Q_i and θ can be found by partially differentiating Eq. 7.1 and equating it to 0. In other words, we determine the following:

$$\frac{\partial L}{\partial Q_j} = 0, \forall j$$

and

$$\frac{\partial L}{\partial \theta} = 0$$

Doing so, we get

$$-\frac{D_j}{Q_j^2} C_{oj} + \frac{1}{2} iC_j + \frac{\theta}{2} C_j = 0$$

or

[1]This is the standard technique to determine the maxima or minima in multivariable calculus.

$$Q_j = \sqrt{\frac{2D_jC_{oj}}{C_j(i+\theta)}} \tag{7.2}$$

Another condition we need to satisfy is

$$\sum_{j=1}^{n} \frac{Q_j}{2} C_j = B \tag{7.3}$$

Substituting the value of Q_j from Eq. 7.2 in Eq. 7.3, we get

$$\sqrt{\frac{2D_jC_{oj}}{C_j(i+\theta)}} \frac{C_j}{2} = B \tag{7.4}$$

Solving for θ, we get

$$\theta = \frac{1}{2B^2}\left(\sum_{j=1}^{n} \sqrt{C_0D_jC_j}\right)^2 - i \tag{7.5}$$

Proportionality Assumption

Consider a situation where the amount of money invested in an item held in inventory is in proportion to the overall budget. In other words, we assume that

$$\frac{C_1}{h_1} = \frac{C_2}{h_2} = \cdots = \frac{C_n}{h_n} = \frac{C}{h}$$

If this assumption is valid, then we can rewrite Eq. 7.2 as (Nahmias 2005)

$$Q_j = \sqrt{\frac{2D_jC_{oj}}{h_j}} \sqrt{\frac{1}{1 + \frac{\theta C}{h}}} \tag{7.6}$$

It should be noted that $h = iC$. There are two terms in Eq. 7.6. The first term is EOQ, while the second one can be set to m, the multiplier. Eq. 7.6 can thus be rewritten as

$$Q_j = \mathrm{EOQ}_j m \tag{7.7}$$

where

$$m = \sqrt{\frac{1}{1 + \frac{\theta C}{h}}} \tag{7.8}$$

The condition that needs to be satisfied is

$$\sum_{j=1}^{n} Q_j C_j = B$$

Substituting the value of Q_j in Eq. 7.7 in the above equation, we get

$$\sum_{j=1}^{n} m\text{EOQ}_j C_j = B$$

or

$$m = \frac{B}{\sum_{j=1}^{n} \text{EOQ}_j C_j} \tag{7.9}$$

It is noticed that the term m is independent of θ. It is easy to compute m since we need only the EOQ and the budget. This method can be only used if the proportionality assumption is met. We will now illustrate the application of the budget constraint model with a numerical example.

Solved Problem 7.1
Inventory parameters for three products – A, B, and C – sold by a retailer are shown in Table 7.3. The retailer uses an inventory carrying rate of 25% per annum. What would be the economic lot size if the retailer does not want to invest more than \$10,000 in the average inventory of these three products?

Solution
Using the standard EOQ formula (Eq. 3.8) we derived in Chap. 3, we can determine the EOQs for the three products:

$$Q_A = \sqrt{\frac{2 \times 1000 \times 30}{0.25 \times 50}} = 69 \text{ units}$$

$$Q_B = \sqrt{\frac{2 \times 1500 \times 35}{0.25 \times 100}} = 65 \text{ units}$$

$$Q_C = \sqrt{\frac{2 \times 2500 \times 50}{0.25 \times 150}} = 82 \text{ units}$$

Table 7.3 Inventory data for Solved Problem 7.1

Parameter	Product A	Product B	Product C
Annual demand	1000	1500	2500
Ordering cost	\$30	\$35	\$50
Unit cost	\$50	\$100	\$150

The average investment in inventory for the above lot sizes can be determined using Eq. 7.3, which is

$$= \frac{1}{2}[(69 \times 50) + (65 \times 100) + (82 \times 150)] = \$11,125$$

It is noticed that the investment in average inventory violates the specified budget constraint of $10,000. Therefore, these lot sizes are not feasible. We need to determine the value of θ to compute feasible lot sizes that satisfy the budgetary constraint. Using Eq. 7.5, we get

$$\theta = \frac{1}{2B^2}\left(\sum_{j=1}^{n} \sqrt{C_0 D_j C_j}\right)^2 - i = 0.058$$

We can now compute the revised EOQ values by substituting the value of θ in Eq. 7.2. We get

$$Q_A = \sqrt{\frac{2 \times 1000 \times 30}{0.308 \times 50}} = 62 \text{ units}$$

$$Q_B = \sqrt{\frac{2 \times 1500 \times 35}{0.308 \times 100}} = 58 \text{ units}$$

$$Q_C = \sqrt{\frac{2 \times 2500 \times 50}{0.308 \times 150}} = 74 \text{ units}$$

The average investment in inventory is

$$= \frac{1}{2}[(62 \times 50) + (58 \times 100) + (74 \times 150)] = \$10,000$$

With the revised EOQs, the investment in average inventory satisfies the specified budgetary constraint. Therefore, the economic and feasible lot sizes for products A, B, and C are 62, 58, and 74 units, respectively.

Solved Problem 7.2

Determine the optimal quantities in Solved Problem 7.1 assuming that the ratio of unit cost of the product to its carrying cost is constant. In other words,

$$\frac{C_A}{h_A} = \frac{C_B}{h_B} = \frac{C_C}{h_C}$$

where C_A, C_B, and C_C are the unit costs and h_A, h_B, h_C are the carrying costs for products A, B, and C, respectively.

Solution
The initial EOQs are 69, 65, and 82 units (see solution to Solved Problem 7.1), respectively, and the average investment in inventory is \$11,125 which violates the budgetary constraint. If proportionality condition is assumed, then we have a simple solution on hand. We do not need to compute the Lagrangean factor. Instead, we would need to determine the multiplier, m, which is given by Eq. 7.9:

$$m = \frac{B}{\sum_{j=1}^{n} \mathrm{EOQ}_j C_j}$$

In this problem, the budgetary constraint B is \$10,000 and $\sum_{j=1}^{n} EOQ_j C_j = \$11,125$. Therefore, we get

$$m = \frac{10000}{11125} = 0.899$$

Multiplying the initial EOQs with m, we get

$$Q_A = 0.899 \times 69 = 62 \text{ units}$$
$$Q_B = 0.899 \times 65 = 58 \text{ units}$$
$$Q_C = 0.899 \times 82 = 74 \text{ units}$$

It is noticed that the solution obtained for this problem is the same as what we obtained in Solved Problem 7.1.

7.3 Space Constraint

Another common constraint firms face is that of storage space. The EOQs (or the Q) computed for a given problem may be huge, and the firm may not have sufficient space to store all the procured items. We need to determine Q for each of the items such that they are not only optimal but also feasible.

Consider an inventory system with n items. Let S be the maximum amount of space that is available to stock the items. If Q_j is the optimal order size, C_{oj} is the ordering cost, D_j is the annual demand, i is the inventory carrying rate per year, and C_j is the unit cost for the jth item in the system, the problem can be formulated as follows:

$$\text{Min} \sum_{j=1}^{n} \left(\frac{D_j}{Q_j} C_{oj} + i \frac{Q_j}{2} C_j \right)$$

subject to the space constraint,

$$\sum_{j=1}^{n} \frac{Q_j C_j}{2} \leq S$$

If the constraint is satisfied by individual optimal order quantities, we not only have the individual optimal Q values but also those that are feasible Q values. However, when the constraint is not satisfied, the individual Q values obtained are infeasible, and we rewrite the constraint as (Srinivasan 2010)

$$\sum_{j=1}^{n} \frac{Q_j S_j}{2} = S \tag{7.10}$$

Just like in the previous section, we introduce a Lagrangean multiplier, θ. The objective now is to find the optimal as well as feasible values for each of j. The Lagrangean function can be written as

$$L = \sum_{i=j}^{n} \frac{D_j}{Q_j} C_{oj} + \sum_{i=j}^{n} i \frac{Q}{2} C_j + \theta \sum_{i=j}^{n} \frac{Q_j S_j}{2} - S \tag{7.11}$$

The optimal values for Q_i and θ can be found by partially differentiating Eq. 7.11 and equating it to 0. In other words, we determine the following:

$$\frac{\partial L}{\partial Q_j} = 0, \forall j$$

and

$$\frac{\partial L}{\partial \theta} = 0$$

Partially differentiating Eq. 7.11 with respect to Q, we get

$$-\frac{D_j}{Q_j^2} C_{oj} + \frac{1}{2} i C_j + \frac{\theta S_j}{2} = 0 \tag{7.12}$$

or

$$Q_j = \sqrt{\frac{2 D_j C_{oj}}{i C_j + \theta S_j}} \tag{7.13}$$

Partially differentiating Eq. 7.11 with respect to θ and equating it to 0, we get

$$\sum_{j=1}^{n} \frac{Q_j S_j}{2} = S \tag{7.14}$$

We will need to use trial and error method to determine the value of θ which will satisfy Eq. 7.14. You may also use GOAL SEEK to determine θ. See Appendix 7A.

Proportionality Assumption

As discussed in the previous section, when the proportionality assumption is satisfied, the multiplier m can be used, which is given by

$$m = \frac{S}{\sum_{j=1}^{n} EOQ_j S_j} \tag{7.15}$$

where S_j is the space consumed and EOQ_j is the optimal order size for jth item. The following example illustrates the application of the space constraint theory.

Solved Problem 7.3

Gaudi is a retailer for overhead water tanks that are available in three different capacities – 2000 liters, 3000 liters, and 4000 liters. Table 7.4 shows the inventory parameters of these water tanks. If Gaudi has a space limitation of 3500 cubic feet in their showroom, determine the optimal order quantities for these water tank models that will satisfy the space constraint.

Solution

To start with, we use Eq. 3.8 we derived in Chap. 3 to determine the EOQs for the three models of water tanks:

$$Q_{2k} = \sqrt{\frac{2 \times 1000 \times 30}{0.20 \times 150}} = 45 \text{ units}$$

$$Q_{3k} = \sqrt{\frac{2 \times 1500 \times 35}{0.20 \times 200}} = 51 \text{ units}$$

$$Q_{4k} = \sqrt{\frac{2 \times 2500 \times 50}{0.20 \times 250}} = 71 \text{ units}$$

Table 7.4 Inventory data for Solved Problem 7.3

Parameter	Q-2 k Liters	Q-3 k Liters	Q-4 k Liters
Annual demand (Nos)	1000	1500	2500
Ordering costs ($/order)	30	35	50
Unit cost ($)	150	200	250
Carrying rate (% per year)	0.20	0.20	0.20
Space required (cubic feet)	27	42	64

The average space for the above lot sizes can be determined from Eq. 7.14, which is

$$= \frac{1}{2}[(45 \times 27) + (51 \times 42) + (71 \times 64)]$$
$$= 3951 \text{ cubic feet}$$

Notice that the space required by these lot sizes violates the space constraint of 3500 cubic feet. Therefore, these lot sizes are not feasible. We need to use Eq. 7.13 to determine the feasible order quantities. Because θ is unknown, we need to use trial and error to compute the feasible lot sizes that satisfy the space constraint. One method would be to compute the lower and upper bound values for θ. The lower and upper bound values can be found by assuming proportional ratios. In other words, we first find m, using Eq. 7.15, which is

$$m = \frac{3500 \times 2}{[(45 \times 27) + (51 \times 42) + (71 \times 64)]} = 0.885$$

We use this multiplier to obtain new EOQs, which are

$$Q_{2k} = 45 \times 0.885 = 40 \text{ units}$$
$$Q_{3k} = 51 \times 0.885 = 45 \text{ units}$$
$$Q_{4k} = 71 \times 0.885 = 63 \text{ units}$$

We can now use these to determine θ values using Eq. 7.13, which are as follows:

$$\theta_{2k} = 0.298$$
$$\theta_{3k} = 0.255$$
$$\theta_{4k} = 0.209$$

The least value is 0.209 and the highest is 0.298. These are the lower and upper bound values for θ. Now it becomes easy for us to use trial and error to determine the Q values that also satisfy the space constraint. Table 7.5 shows the Q values for different values of θ.

Table 7.5 Values for θ and optimal lot sizes

θ	Q-2 k	Q-3 k	Q-4 k	Total space (Cubic feet)
0.3	39.7	44.7	60.1	3398
0.2	41.2	46.6	63.1	3554
0.23	40.7	46	62.2	3506
0.234	40.6	45.9	62	3496

Table 7.6 Inventory data for Solved Problem 7.4

Parameter	Product X	Product Y
Annual demand (Nos)	10,000	15,000
Ordering costs ($/order)	300	350
Unit cost ($)	100	80
Carrying rate (% per year)	0.25	0.25
Space required (cubic feet)	50	125

Notice that for $\theta = 0.234$, the values of Q for the three models satisfy the space requirements. The optimal Q values are 41, 46, and 62, respectively for the models, and the average space requirement is 3496 cubic feet which satisfies the specified space constraint of 3500 cubic feet.

Another method to determine θ would be to use the GOAL SEEK function available in MS Excel. The procedure to use this function is explained in Appendix 7A.

Solved Problem 7.4

Determine the optimal and feasible order quantities for the following two products if there is a space restriction of 40,000 cubic feet (Table 7.6).

Solution

We start with using Eq. 3.8 to determine the EOQs for the two products:

$$Q_x = \sqrt{\frac{2 \times 10000 \times 300}{0.25 \times 100}} = 490 \text{ units}$$

$$Q_y = \sqrt{\frac{2 \times 15000 \times 350}{0.25 \times 80}} = 725 \text{ units}$$

The average space for the above lot sizes can be determined from Eq. 7.14, which is

$$= \frac{1}{2}[(490 \times 50) + (725 \times 125)]$$
$$= 57533 \text{ cubic feet}$$

The average space required by these products violates the space constraint of 40,000 cubic feet. Therefore, these lot sizes are not feasible. We need to use Eq. 7.13 to determine the feasible order quantities. Since θ is unknown we need to first compute the lower and upper bound values for θ. The lower and upper bound values can be found by assuming proportional ratios. In other words, we first find m, using Eq. 7.15, which is

$$m = \frac{40000 \times 2}{[(490 \times 50) + (725 \times 125)]} = 0.695$$

Table 7.7 Different values of θ Theta

θ	Q-x	Q-y	Total space (Cubic feet)
0.53	341	349	30,340
0.30	387	427	36,390
0.20	414	483	40,541
0.21	411	476	40,057
0.2113	410	476	39,996

We use this multiplier to obtain new EOQs, which are

$$Q_x = 490 \times 0.695 = 341 \text{ units}$$
$$Q_y = 725 \times 0.695 = 504 \text{ units}$$

The next step is to use these EOQs to determine θ values using Eq. 7.13, which are as follows:

$$\theta_x = 0.534$$
$$\theta_y = 0.171$$

These are the lower and upper bound values for θ. We can now use different values of θ between 0.534 and 0.171 to determine the Q values that also satisfy the space constraint. Table 7.7 shows the Q values for different values of θ.

It is noticed that for $\theta = 0.2113$, the values of Q for the two products satisfy the space requirements. The optimal Q values are 410 and 476, respectively, and the average space requirement is 39,996 cubic feet which satisfies the specified space constraint of 40,000 cubic feet.

7.4 Number of Orders Constraint

Let us consider an inventory system with n items. The total ordering cost would depend on the number of orders being placed. To reduce the ordering cost, one may want to place orders a limited number of times. Let N be the maximum number of orders that can be placed. If Q_j is the optimal order size, C_{oj} is the ordering cost, D_j is the annual demand, i is the inventory carrying rate per year, and C_j is the unit cost for the jth item in the system, the problem can be formulated as follows:

$$\text{Min} \sum_{j=1}^{n} \left(\frac{D_j}{Q_j} C_{oj} + i \frac{Q_j}{2} C_j \right)$$

subject to

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} \leq N$$

The objective function measures the total inventory costs while the term in the constraint measures the number of orders placed. If the constraint is satisfied by individual optimal order quantities, we not only have the individual optimal Q values but also those that are feasible Q values. However, when the constraint is not satisfied the individual Q values obtained are infeasible, and we rewrite the constraint by replacing the inequality as

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = N$$

Just like in the previous section, we introduce a Lagrangean multiplier, θ, to solve this nonlinear objective function The objective now is to find the optimal as well as feasible values for each of j. The Lagrangean function can be written as

$$L = \sum_{i=j}^{n} \frac{D_j}{Q_j} C_{oj} + \sum_{i=j}^{n} i \frac{Q}{2} C_j + \theta \left(\sum_{j=1}^{n} \frac{D_j}{Q_j} - N \right) \tag{7.16}$$

The optimal values for Q_i and θ can be found by partially differentiating Eq. 7.16 and equating it to 0. In other words, we determine the following:

$$\frac{\partial L}{\partial Q_j} = 0, \forall j$$

and

$$\frac{\partial L}{\partial \theta} = 0$$

First, we partially differentiate L with respect to Q. By doing so, we get

$$-\frac{D_j}{Q_j^2} C_{oj} + \frac{1}{2} i C_j - \theta \frac{D_j}{Q_j^2} = 0$$

or

$$Q_j = \sqrt{\frac{2D_j \left(C_{oj} + \theta \right)}{iC_j}} \tag{7.17}$$

Another condition we need to satisfy is

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = N \tag{7.18}$$

Substituting the value of Q_j from Eq. 7.2 in Eq. 7.3, we get

$$\sum_{j=1}^{n} \frac{D_j\sqrt{iC_j}}{\sqrt{2D_j(C_{oj}+\theta)}} = N$$

Solving for θ, we get

$$\theta = \frac{i\left(\sum_{j=1}^{n}\sqrt{D_jC_j}\right)^2}{2N^2} - C_{oj}$$

If

$$C_{o1} = C_{02} = \cdots = C_o$$

we have

$$\theta = \frac{i\left(\sum_{j=1}^{n}\sqrt{D_jC_j}\right)^2}{2N^2} - C_o \tag{7.19}$$

Solved Problem 7.5
Table 7.8 shows the inventory data for three products that are sold by a retail firm. The senior management of the firm is concerned about high ordering costs, and to contain it has imposed a restriction on the number of orders to 20 per year. Compute the optimal and feasible order quantities if the ordering cost is \$200 per order.

Table 7.8 Inventory data for Solved Problem 7.5

Parameter	Product A	Product B	Product C
Annual demand (Nos)	1000	1500	2500
Unit cost (\$)	100	80	60
Carrying rate (% per year)	0.30	0.30	0.30

Solution
The first step is to find the EOQs for each product using Eq. 3.8:

$$Q_{2k} = \sqrt{\frac{2 \times 1000 \times 200}{0.30 \times 100}} = 115 \text{ units}$$

$$Q_{3k} = \sqrt{\frac{2 \times 1500 \times 200}{0.30 \times 80}} = 158 \text{ units}$$

$$Q_{4k} = \sqrt{\frac{2 \times 2500 \times 200}{0.30 \times 60}} = 236 \text{ units}$$

The number of orders required to be placed for the above lot sizes can be calculated using Eq. 7.18. We get

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = 8.7 + 9.5 + 10.6 = 28.7 \text{ orders per year}$$

This is greater than the restriction of 20 orders per year. We can use Eq. 7.19 to compute the Lagrangean multiplier, θ, which in this case is 213.4. Substituting this, we can determine the optimal value of lot sizes by substituting the value of θ in Eq. 7.17, as follows:

$$Q_{2k} = \sqrt{\frac{2 \times 1000 \times (200 + 213.4)}{0.30 \times 100}} = 166 \text{ units}$$

$$Q_{3k} = \sqrt{\frac{2 \times 1500 \times (200 + 213.4)}{0.30 \times 80}} = 227 \text{ units}$$

$$Q_{2k} = \sqrt{\frac{2 \times 2500 \times (200 + 213.4)}{0.30 \times 60}} = 339 \text{ units}$$

The optimal values are 166, 227, and 339 units for products A, B, and C, respectively. Using Eq. 7.18 with the new Q values, we get the total number of orders to be placed for the above lot sizes as

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = 6 + 6.6 + 7.4 = 20 \text{ orders per year}$$

which is satisfying the number of orders constraint. These Q values are optimal as well as feasible.

Solved Problem 7.6
Consider two products X and Y whose inventory data are shown in Table 7.9. If the number of orders is restricted to 48 per year, compute the optimal and feasible order quantities if the ordering cost is $50 per order.

Table 7.9 Inventory data for Solved Problem 7.6

Parameter	Product X	Product Y
Annual demand (Nos)	10,000	15,000
Unit cost ($)	35	45
Carrying rate (% per year)	0.25	0.25

Solution
The optimal order quantities can be found using Eq. 3.8, which are

$$Q_x = \sqrt{\frac{2 \times 10000 \times 50}{0.25 \times 35}} = 338 \text{ units}$$
$$Q_y = \sqrt{\frac{2 \times 15000 \times 50}{0.25 \times 45}} = 365 \text{ units}$$

The number of orders required to be placed for the above lot sizes can be calculated using Eq. 7.18. We get

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = 29.6 + 41.1 = 70 \text{ orders per year}$$

This is greater than the restriction of 48 orders per year. We can use Eq. 7.19 to compute the Lagrangean multiplier, θ, which is 58.35. Substituting this value in Eq. 7.17, we can determine the optimal value of lot sizes, as follows:

$$Q_x = \sqrt{\frac{2 \times 10000 \times (50 + 58.35)}{0.25 \times 35}} = 497 \text{ units}$$
$$Q_y = \sqrt{\frac{2 \times 15000 \times (50 + 58.35)}{0.25 \times 45}} = 538 \text{ units}$$

The optimal values are 497 and 538 units for products X and Y, respectively. Using Eq. 7.18 with the new Q values, we get the total number of orders, which is

$$\sum_{j=1}^{n} \frac{D_j}{Q_j} = 20.12 + 27.88 = 48 \text{ orders per year}$$

These Q values are optimal as well as feasible since they satisfy the number of constraints.

7.5 Multiple Constraints

In this section, we will discuss solution approach to problems that are subject to more than one constraint. We start by taking up one constraint and obtain optimal values. We then take up the second constraint and check if the same optimal values satisfy the second constraint as well. If they do, then we have an optimal solution that satisfies both the constraints. It is important that the constraints in a problem are not opposing each other. Let us consider the following scenarios for a given inventory problem.

7.5.1 Budgetary and Number of Orders Constraint

Consider a problem which is subject to both a budgetary constraint as well as number of orders constraint. Solution to the budgetary constraint ($\sum QC$) will attempt to reduce the lot sizes, while the same for number of orders constraint $\left(\frac{D}{Q}\right)$ will try to increase the lot sizes. Thus, it becomes difficult to obtain a solution that satisfies both constraints. In such situations, it is best to satisfy the most important constraint rather than both.

7.5.2 Space and Number of Orders Constraint

Consider a case where we have both space constraint as well as number of orders constraint. While the solution to a space constraint will try to reduce the lot sizes, the number of orders constraint will try to increase the lot sizes. It is difficult to find a feasible solution that satisfies both. Again, in this situation, we need to attempt to obtain a solution to the most important constraint.

7.5.3 Budgetary and Space Constraint

Let us now consider a problem that is subject to budgetary as well as space constraint. Solution to the budgetary constraint ($\sum QC$) will attempt to reduce the lot sizes. The solution to a space constraint will also try to reduce the lot sizes. Thus, it is easy to obtain a solution that satisfies both constraints.

Solved Problem 7.7
Inventory parameters for three products are as shown in Table 7.10. Determine the optimal lot sizes for these products that are subject to the following constraints:

- Average value of inventory held must not exceed \$8000.
- Space occupied by the stock must not exceed 6000 cubic feet.

Table 7.10 Inventory data for Solved Problem 7.7

Parameter	Product A	Product B	Product C
Annual demand (Nos)	1000	1500	2500
Unit cost ($)	100	80	60
Ordering cost ($ per order)	30	35	65
Carrying rate (% per year)	0.30	0.30	0.30
Space consumed (cubic feet)	20	30	80

Solution

Let us first solve the problem for the inventory restriction. The initial EOQ values for the three products are

$$Q_A = \sqrt{\frac{2 \times 1000 \times 30}{0.3 \times 100}} = 45 \text{ units}$$

$$Q_B = \sqrt{\frac{2 \times 1500 \times 35}{0.3 \times 80}} = 66 \text{ units}$$

$$Q_C = \sqrt{\frac{2 \times 2500 \times 65}{0.3 \times 60}} = 134 \text{ units}$$

The average investment in inventory for the above lot sizes can be determined from Eq. 7.3, which is

$$= \frac{1}{2}[(45 \times 100) + (66 \times 80) + (134 \times 60)] = \$8,913$$

Notice that the investment in inventory violates the specified budget constraint of $8000. Therefore, these lot sizes are not feasible. We need to determine the value of θ to compute feasible lot sizes that satisfy the budgetary constraint. Using Eq. 7.5, we get $\theta = 0.072$. By substituting the value of θ in Eq. 7.2, we can now compute the revised EOQ values. We get

$$Q_A = \sqrt{\frac{2 \times 1000 \times 30}{0.372 \times 100}} = 40 \text{ units}$$

$$Q_B = \sqrt{\frac{2 \times 1500 \times 35}{0.372 \times 80}} = 59 \text{ units}$$

$$Q_C = \sqrt{\frac{2 \times 2500 \times 65}{0.372 \times 60}} = 121 \text{ units}$$

The average investment in inventory is

$$= \frac{1}{2}[(40 \times 100) + (59 \times 80) + (121 \times 60)] = \$8,000$$

Table 7.11 Values for θ and optimal lot sizes

θ	Product A	Product B	Product C	Total space (Cubic feet)
0.015	39.97	58.92	117.49	5983
0.014	39.99	58.95	117.69	5992
0.013	40.00	58.98	117.89	6001
0.0131	40.00	58.98	117.87	6000

We now notice that the investment in inventory satisfies the specified budgetary constraint. Therefore, the economic and feasible lot sizes for products A, B, and C are 40, 59, and 121 units, respectively. Next, we solve the problem for the space constraint. The space requirement for the above lot sizes is.

$$= \frac{1}{2}[(40 \times 20) + (59 \times 30) + (121 \times 80)] = 6125 \text{ cubic feet.}$$

The lot sizes we determined do not satisfy the space constraint. We therefore need to find the value of θ that would help us compute the optimal and feasible lot sizes. We need to use trial and error values for θ from Eq. 7.9 to compute the feasible lot sizes that satisfy the space constraint. Table 7.11 shows the values of lot sizes for different values of θ.

We can see from Table 7.6 that the space constraint is met for $\theta = 0.0131$. The lot sizes corresponding to this value are 40, 59, and 118 units for products A, B, and C, respectively. These lot sizes meet both the budgetary as well as space constraints.

7.6 Coordinated Replenishment

Considering the fact that inventory decisions on items in a system interact with each other, another topic of interest to inventory managers is that of coordinated replenishment, also referred to as joint ordering. When many items are ordered from the same supplier, or when items share the same mode of transportation, it makes economic sense to coordinate their replenishment. By ordering many items jointly in one order, managers can take advantage of economies of scale and can help reduce the overall inventory costs (Silver et al. 1998; Vrat 2014).

7.6.1 Costs in Coordinated Replenishment

Two types of costs[2] are incurred in coordinated replenishment process:

[2]These costs are referred as the Major Cost and the Minor cost (or line cost) by Silver et al. (1998).

- A fixed cost of ordering an item. This cost is independent of the number of items being procured. This cost is represented by C_o.
- A shared cost, that is, if two items that are ordered jointly are shipped together, then these two items share the cost of transportation. This cost is represented by C_t. (This can also be considered as the cost of adding a line to an order, while raising an order.)

7.6.2 Assumptions

Following are some of the assumptions made in the treatment of this problem:

- There are no resource constraints.
- A fixed ordering cost, C_o, is incurred every time an order is placed for an item. There is an additional cost, C_t, for every order. This could be the transportation cost, for example. If, therefore, only one item is ordered, the total ordering cost would be $C_o + C_t$.
- Demand for the items is deterministic
- All items have the *same order frequency* and are received together.
- The unit cost for each item is not dependent on the quantity ordered. In other words, no quantity discounts are offered.

Let us consider an inventory system with n items. C_o is the ordering cost per order. If the supplier decides to fill the order by sending all n items together, then the total cost of transportation, C_t, is shared by these n items. The total cost of the order is therefore

$$nC_o + C_t \tag{7.20}$$

If the demand for the jth item is D_j and the order quantity is Q_j, the total number of orders to be placed, k, would be

$$k = \frac{D_j}{Q_j} \tag{7.21}$$

The order size Q_j can be written as

$$Q_j = \frac{D_j}{k} \tag{7.22}$$

If the inventory carrying rate is i and the unit cost for the jth item is C_j, the total inventory costs would be

$$\text{TIC} = k(nC_o + C_t) + i\sum_{j=1}^{n}\frac{D_j}{2k}C_j \tag{7.23}$$

Differentiating Eq. 7.23 with respect to k and equating it to 0, we get

$$(nC_o + C_t) - i\sum_{j=1}^{n} \frac{D_j}{2k^2} C_j = 0$$

or

$$k = \sqrt{\frac{i\sum_{j=1}^{n} D_j C_j}{2(nC_o + C_t)}} \tag{7.24}$$

Eq. 7.24 indicates the number of orders that needs to be placed to minimize the total inventory costs. We will now demonstrate the application of this theory with a numerical problem.

Solved Problem 7.8
Table 7.12 shows the inventory parameters for raw materials that a retail firm sources from its local supplier. Until recently, the firm was ordering raw materials separately. However, it has now decided to place a joint order so that all raw materials arrive together. The fixed cost per order is \$10 for each item. If the firm incurs a transportation cost of \$500, compute the optimal number of orders that needs to be placed to minimize the total inventory costs. Also compute the savings if the firm was ordering separately.

Solution

Items ordered jointly
We use Eq. 7.24 to determine the value of number of orders in the case of joint ordering of the three raw materials. To start with, we compute the numerator, which is

$$\sum_{j=1}^{n} iD_j C_j = 0.3 \times [(1000 \times 100) + (1500 \times 80) + (2500 \times 60)] = 111,000$$

The total cost of an order is

$$nC_{oj} + C_t = 30 + 500 = 530$$

Table 7.12 Inventory data for Solved Problem 7.8

Parameter	Raw material A	Raw material B	Raw material C
Annual demand (Nos)	1000	1500	2500
Unit cost (\$)	100	80	60
Carrying rate (% per year)	0.30	0.30	0.30

Substituting the above values in Eq. 7.24, we get

$$k = \sqrt{\frac{111000}{2 \times (530)}} = 10.2$$

The total inventory cost can be determined using Eq. 7.23 . Substituting the values, we get

$$\text{TIC} = (10.2 \times 530) \times \frac{111000}{(2 \times 10.2)} = \$10,847$$

The optimal number of joint orders is 10.2, and the total inventory cost is $10,847.

Items ordered separately
The total inventory costs for raw materials A, B, and C would be as follows:

Step 1: Compute EOQs. Since the raw materials are being ordered separately, each order would incur an ordering cost of $510 for each item		
$Q_A = \sqrt{\frac{2 \times 1000 \times 510}{0.3 \times 100}}$ Or $Q_A = 184$	$Q_B = \sqrt{\frac{2 \times 1500 \times 510}{0.3 \times 80}}$ Or $Q_B = 252$	$Q_C = \sqrt{\frac{2 \times 2500 \times 510}{0.3 \times 60}}$ Or $Q_C = 376$
Step 2: Compute number of orders		
$N_A = \frac{1000}{184} = 5.43$	$N_B = \frac{1500}{252} = 5.95$	$N_A = \frac{2500}{376} = 6.64$
Step 3: Compute ordering costs		
$= 5.43 \times 510$ $= 2769$	$= 5.9 \times 510$ $= 3035$	$= 6.64 \times 510$ $= 3386$
Total ordering cost $= 2769 + 3035 + 3386 = 9190$		
Step 4: Computing carrying costs		
$= \frac{184}{2} \times 0.3 \times 100$ $= 2760$	$= \frac{252}{2} \times 0.3 \times 80$ $= 3024$	$\frac{376}{2} \times 0.3 \times 60$ $= 3384$
Total ordering cost $= 2760 + 3024 + 3384 = 9168$		
TIC $= 9190 + 9168 = \$18,358$		

Total Savings = $18358 – $10847 = $7,511.

7.6.3 *Coordinated Replenishments: Unequal Number of Orders*

One of the assumptions made while deriving the equation for coordinated replenishment was that the number of orders placed for all items is the same. This means that *all* items are ordered together, a situation that may not find many real-life applications. This assumption can be relaxed; however, in such a case, we must find the frequency of placing an order for each of the items, and order those items together whose frequencies are equal. Srinivasan (2010) has shown that in such a situation the number of orders for the *j*th item can be found by

$$m_j = C_x\sqrt{\frac{C_t}{D_jC_j}}$$

where C_x is a constant given by

$$C_x = \sqrt{\frac{D_jC_j}{C_t + C_o}}$$

7.7 Summary

In this chapter, we discussed methods of managing inventory of multiple items that are subject to one or more resource constraints. One key takeaway from this chapter is that when the ratio of unit cost of an item (or space consumed by an item) to the carrying cost is constant, a simple solution in the form of a multiplier can be adopted. However, when the assumption is not met, a Lagrangean multiplier (factor) would need to be determined, in some cases by trial and error, to obtain an optimal and feasible economic lot size solution.

We also reviewed methods to solve problems that are subject to more than one constraint. A feasible solution is possible only when revisions made to lot size for both constraints move the decision variable in the same direction.

7.8 Case Study: Joint Replenishment

Hector is a Vice President – Procurements, with FreskoJugo, a company in Houston, Texas, producing fresh fruit juices. Hector is busy in his office trying to understand the inventory costs when Fernanda, his secretary walks in.

"Jerry and Tim from Fruit-o-Vision are here to see you. They have an appointment with you," announces Fernanda.

"Oh yes, please send them in," says Hector.

Two people walk in to Hector's office.

"I am Jerry," says Jerry, introducing himself to Hector, "and this is my colleague Tim. We are from Fruit-o-Vision," adds Jerry.

"We supply fresh fruits to food product manufacturers like yours. We have our own orchards in the Silao and Queretaro in central part of Mexico, and we grow 15 different varieties of fruits including peaches, grapes, mango, avocado, bananas, pineapple, strawberry, green apples, mandarins and several others," says Jerry.

"We do not use any chemicals in our orchards. We use organic manure," Tim intervenes. "We are carrying some samples of our fresh produce which you may want to see."

Tim then shows samples of the fruits grown in their orchard.

"We know your firm is always looking at enlisting fresh fruit suppliers, and we would certainly want to explore doing business with your firm," says Jerry.

"I am impressed with the samples. I am going to get our tasters let me know what they are like," says Hector.

Continuing the discussion, Hector says "I have heard of your orchards but our existing suppliers have been working with us for the last 15 years, and they have been doing a good job for us."

"We source mangoes from Tijuana while we get our supplies of grapes from California, and mandarins from Monterrey," adds Hector.

"One thing I must point out you is that we supply *all* the fruits," says Jerry, emphasizing on the all bit. "This would be a great advantage for you to lower your ordering costs if your firm can place coordinated orders," Jerry adds.

"Hmm. that sounds interesting. I have a meeting with our President later today, and I will discuss this with him and then let us see things pan out. I certainly will keep in touch with you guys, and thanks for visiting FreskoJugo," says Hector as Jerry and Tim prepare to take leave of Hector.

Later that day, Hector is attending a high-level meeting chaired by Issac, the company President.

Addressing the attendees of the meeting, Issac says "I want all of you to focus your attention on this section of our balance sheet under Current Assets. Our inventory – including raw materials, work-in-progress and finished goods – has increased more than 15% over the last year."

"This reflects the amount of money locked up in inventories. Some financial analysts may say this is good since the inventory is an asset. But in my opinion we need to control it."

"While too little inventory can result in lost sales, too much inventory can impact our cash flow problems. With the price of diesel going up each month we can be sure our inventories will definitely increase."

Finishing his presentation, Issac adds "I want each one of you to start thinking about how we could reduce our inventory cost, without impacting our sales, of course."

Hector intervenes "Issac, one of our problems is with our procurement policy. We procure fruits from different vendors – 5 vendors for 5 fruits, and the fixed cost of ordering per order is $40, and it costs us another $80 in shipment. In all $120 per order."

"But what has that got to do with the increase in our inventory costs?"asks Issac.

"The transportation cost is huge, and if we are able to procure all our input materials from one vendor, we would be able to save on our ordering costs using the concept of joint replenishment," argues Hector.

"But that would mean we put all our eggs in the same basket," states Amy, Risk Manager at FreskoJugo.

"Yes, I agree, Amy, there are risks, but we need to analyze if the savings in coordinated orders outweigh those risks," retorts Hector.

"So, Hector and Amy, why don't you both analyze the coordinated supply scenario and let us know next week what savings it would bring us?" asks Issac.

"Will do Issac," says Hector as he prepares to leave the conference room.

Back in his office, Hector types out an email for Jerry.

Hi Jerry,
Thank you and Tim for visiting my office yesterday. I have a small task for you guys. I was looking at placing an order for the items listed below. Before I confirm the order can you please let me know your best unit price and also the freight costs?

S. No.	Item name	Specifications	Quantity (kg)
1	Mango	Ripe yellow, standard size	2000
2	Apple	Green	2500
3	Grapes	Red/Brown	2000
4	Mandarins	Standard size	4400
5	Avocado	Ripe	3100

Would appreciate if you can revert by morning tomorrow.
Warm Regards
Hector

The next morning Hector has an email response from Jerry.

Hi Hector,
Thank you for your email. Please find our best price for the items that you wish to place an order with us for.
Do let me know if you have any questions, and we look forward to working with FreskJugo.

S. No.	Item name	Specifications	Quantity (kg)	Unit price
1	Mango	Ripe yellow, standard size	2000	$2.50
2	Apple	Green	2500	$2.25
3	Grapes	Red/Brown	2000	$3.35
4	Mandarins	Standard size	4400	$1.25
5	Avocado	Ripe	3100	$0.85

With regards to freight, we will ship all the above in one vehicle, and it would cost you $80 per shipment.
Warm Regards
Jerry

Case Study Questions

1. Given the information, compute the cost savings if FreskoJugo places a joint order for all the five items with one supplier, over procuring the same five fruits separately. Assume equal number of orders.
2. Discuss the risks involved in placing orders with one vendor for all their supplies.

7.9 Practice Problems

Problem 7.1

A retailer sells three products – A, B, and C – details of which are shown in Table 7.13. Assume a carrying rate of 30% per year.

What would be the economic lot size if the retailer does not want to invest more than $10,000 in these three products at one time?

What would be the lot size if you assume the ratio of unit cost of the product to its carrying cost is constant?

Answer:

The feasible EOQs that satisfy the budget constraint are 137, 213, and 675 units. The multiplier is 0.624, and θ is 0.468.

Problem 7.2

Table 7.14 shows the inventory parameters of three products that are sold by a retail firm.

What would be the economic lot size if the firm has a storage constraint of 18,000 cubic feet?

Answer

The feasible EOQs that satisfy the space constraint are 216, 327, and 910 units. θ is 0.0409.

Table 7.13 Inventory data for Problem 7.1

Parameter	Product A	Product B	Product C
Annual demand	12,000	15,000	35,000
Ordering cost	$30	$35	$50
Unit cost ($)	50	30	10

Table 7.14 Inventory data for Problem 7.2

Parameter	Product A	Product B	Product C
Annual demand	12,000	15,000	35,000
Ordering cost	$30	$35	$50
Unit cost ($)	50	30	10
Space required (cubic feet)	10	20	30

Table 7.15 Inventory parameters for Solved Problem

Parameter	Product A	Product B	Product C
Annual demand (Nos)	450	325	750
Unit cost ($)	32	12	18
Carrying rate (% per year)	0.25	0.25	0.25

Table 7.16 Inventory parameters for Solved Problem 7.4

Parameter	Strawberry	Mango	Pineapple
Annual demand (Nos)	200	160	800
Unit cost ($)	21	20	65
Ordering cost ($ per order)	12	20	18
Carrying rate (% per year)	0.30	0.30	0.30
Space consumed (cubic feet)	1.15	1.25	2.00

Problem 7.3
Table 7.15 shows the inventory parameters of three products that are sold by a retail firm. The senior management of the firm is concerned about high ordering costs, and in order to contain it has imposed a restriction on the number of orders to 12 per year. Compute the optimal and feasible order quantities if the ordering cost is $50 per order.

Answer
The feasible EOQs that satisfy the number of orders constraint are 93, 130, and 163 units.

Problem 7.4
A fruit vendor sells three types of fruits – strawberry, mango, and pineapple, stocking parameters of which are as shown in Table 7.16. Determine the optimal lot sizes for these fruits that are subject to the following constraints:

- Average value of inventory held must not exceed $1600.
- Space occupied by the stock must not exceed 60 cubic feet.

Answer
The EOQs are 21, 24, and 33 units that meet both the budget and space constraints.

Appendix 7A: Using GOAL SEEK to Determine Lagrangean Multiplier

An easier method to find θ, the Lagrangean multiplier, is to use the GOAL SEEK function in MS Excel. The procedure for (explained with reference to space constraint, with three products) this is as follows. Set up an excel spreadsheet with the columns as shown in Fig. 7.1.

Fig. 7.1 Input data setup

	J	K	L	M
7		Q-2k	Q-3k	Q-4k
8	Demand	1000	1500	2500
9	Order Cost	30	35	50
10	Unit Cost	150	200	250
11	Space	27	42	64
12	Carrying Rate	0.2	0.2	0.2

	J	K	L	M	N
15	EOQ	45	51	71	
16	Total Space	1215	2142	4544	3950.5
17	Space Constraint				3500

Fig. 7.2 GOAL SEEK – Computing EOQ and space requirement

Next, create the following rows to compute the EOQ and the space requirement. The formula to compute EOQ for each of the three products is as given below:

Cell	Formula
K15	Round(SQRT((2*K8*K9)/(K10*K12)),0)
L15	Round(SQRT((2*L8*L9)/(L10*L12)),0)
M15	Round(SQRT((2*M8*M9)/(M10*M12)),0)

The formulae to compute the space requirement for each of the items are shown in Fig. 7.2. The formulae for these are as given below:

Cell	Formula
K16	K15*K11
L16	L15*L11
M16	M15*M11

We are now ready to set up the GOAL SEEK area which needs to be set up as shown in Fig. 7.3.

The formulae to be used in the GOAL SEEK area are given below. The last one is the total space requirement for all three products put together

Cell	Formula
K20	SQRT((2*K8*K9)/((K12*K10) + ($N20*$K$11)))
L20	SQRT((2*L8*L9)/((L12*L10) + ($N20*$L$11)))
M20	SQRT((2*M8*M9)/((M12*M10) + ($N20*$M$11)))
O20	((K20*K$11) + (L20*L$11) + (M20*M$11))/2

The GOAL SEEK function can be used as follows. Navigate to Data > What-if-Analysis > Goalseek in MS Excel when a GOAL SEEK window pops up. In the

	J	K	L	M	N	O
19	GOALSEEK	Q-2k	Q-3k	Q-4k	θ	Total Space (Cubic Feet)
20		40.6681612	45.9335391	62.0742363	0.23251493	3500.000061
21	Revised EOQ (Rounded)	41	46	62		

Fig. 7.3 GOAL SEEK – Setting constraint

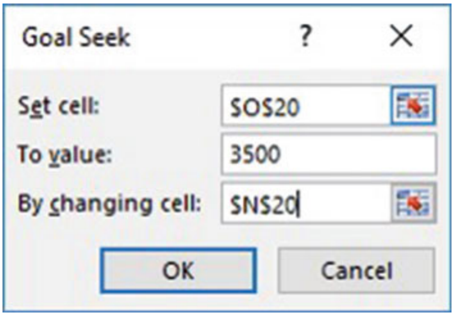

Fig. 7.4 GOAL SEEK window

pop-up window, set the cell to O$20 (space constraint) and the value to 3500 (cubic feet) by changing the cell which has the θ Theta value. This is shown in Fig. 7.4

The θ theta value (Lagrangean multiplier) that satisfies the constraint is displayed in cell N20.

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Chapter 8
Selective Inventory Control Models

8.1 Need for Selective Inventory Control

Monitoring and controlling of inventory items is very expensive. In a multiproduct inventory system, all items held in stock are not equally profitable. An item with a purchase price of $200 may be considered high-value, but may be used sparingly. The purchase price of another item may be $0.50, which is considered relatively low-value. However, this item may be used in large quantities. Table 8.2 shows a sample of 16 items used by Rosetta's. Which of items would the inventory manager focus on more?
In this chapter, we address multi-item inventory situation which allows inventory managers to focus more on high-value, high-usage items than on all items equally.

Inventory managers managing large number of items would need a better method of management. It is important that the inventory managers focus more on controlling high-value, high-usage items than on all items equally. Several techniques have been used by inventory managers, and almost all of these techniques classify items held in inventory into three or more categories. This enables inventory managers apply requisite control that is justified for each class of items. Some of the popular techniques used in practice include:

- ABC classification
- VED classification
- FSN classification

Each of the above is discussed in some detail in this chapter.

D. Shenoy, R. Rosas, *Problems & Solutions in Inventory Management*,
DOI 10.1007/978-3-319-65696-0_8

8.2 ABC Classification[1]

ABC classification is a technique that helps inventory managers perform selective inventory control by focusing on high-value, high-usage items than other items in the inventory system. This technique is based on Pareto's theory, which when adapted to inventory management can be summarized as follows:

> 20% of the items in an inventory system account for 80% of dollar-volume sales, the next 30% account for 15% of dollar-volume sales, while the last 50% of the items account for 5% of dollar-volume sales.

Many organizations use the ABC classification technique to manage their inventory – exercise tight control over a small number (usually up to 20% of items in stock) of high-value items, moderate control over a larger number (about 30% of items in stock) of moderately expensive items, and simple control over a very large number of items (about 50% of items in stock) of low-value items (Table 8.1).

Table 8.2 shows items at Rosetta's, along with unit price and annual demand. Let us use the ABC analysis to determine the level of control the inventory manager needs to apply for each of these items.

The first step in ABC analysis is to compute the annual usage value for each of the items. This can be obtained by multiplying the unit price of the item with its annual demand. This is shown Table 8.3.

The next step is to rank-order the items based on their annual usage value, in descending order. Table 8.4 below shows the rank-ordered items.

Figure 8.1 shows the ABC classification chart. Note that 3 out of 16 items (approximately 19% of items) – corn flour, cooking gas, and vegetable oil – contribute to roughly 80% of the inventory usage value. The next five items (close to 30% of the items) – cornmeal additive, cashew nut paste, vanilla essence, butter, and liqueur – contribute to roughly about 15% of the inventory value. The last eight items (or 50% of the items) contribute to the remaining 5% of the inventory usage value.

Table 8.1 ABC classification strategy

Category	Quantity	Value of items	Control
A class	20% of items	80% inventory value	Tight
B class	30% of items	15% inventory value	Moderate
C class	50% of items	5% inventory value	Simple

[1]Also called ABC Analysis.

Table 8.2 Sample of 16 items at Rosettas

S. No.	Item name	Unit price ($)	Unit	Annual demand
1	Vegetable oil	20	Liter	7200
2	Corn flour	30	Kilogram	25,000
3	Butter	65	Kilogram	500
4	Water	0.5	Liter	25,000
5	Cornmeal additive	70	Kilogram	750
6	Salt	6	Kilogram	100
7	Vanilla essence	300	Milliliter	120
8	Eggs	25	Dozens	240
9	Cashew nut paste	400	Jar	120
10	Printed polybag	8	100 numbers	1200
11	Yeast (imported)	200	Pounds	100
12	Pepper	300	Kilogram	36
13	Goat cheese	500	Kilogram	24
14	Liqueur	200	Liter	120
15	Cooking soda	45	Kilogram	100
16	Cooking gas	4600	Full tank (300 liters)	48

Table 8.3 Computing annual usage value

S. No.	Item name	Unit price ($)	Annual demand	Annual usage value
1	Vegetable oil	20	7200	144,000
2	Corn flour	30	25,000	750,000
3	Butter	65	500	32,500
4	Water	0.5	25,000	12,500
5	Cornmeal additive	70	750	52,500
6	Salt	6	100	600
7	Vanilla essence	300	120	36,000
8	Eggs	25	240	6000
9	Cashew nut paste	400	120	48,000
10	Printed polybag	8	1200	9600
11	Yeast (imported)	200	100	20,000
12	Pepper	300	36	10,800
13	Goat cheese	500	24	12,000
14	Liqueur	200	120	24,000
15	Cooking soda	45	100	4500
16	Cooking gas	4600	48	220,800

ABC classification method is probably the simplest of the selective inventory control methods to implement and use. A great advantage of this method is that it allows inventory managers to manage their review time wisely – they are able to spend more effort monitoring items that have the biggest impact on the cost and usage, and therefore, using this method would enable managers apply closer and

Table 8.4 Rank-ordering by annual usage value

Item Code	Item Name	Annual Usage Value	Proportion of usage value	Cumulative proportion	Inventory Class	
2	Corn Flour	750000	54.20%	54.20%	A	
16	Cooking Gas	220800	15.96%	70.15%	A	Top 3 items (or 20% of the items) contribute to 80% of total inventory usage value.
1	Vegetable Oil	144000	10.41%	80.56%	A	
6	Cornmeal Additive	52500	3.79%	84.35%	B	
9	Cashew-nut Paste	48000	3.47%	87.82%	B	Next 5 items (or 30% of the items) contribute to 15% of total inventory usage value.
7	Vanilla Essence	36000	2.60%	90.42%	B	
3	Butter	32500	2.35%	92.77%	B	
14	Liqueur	24000	1.73%	94.51%	B	
11	Yeast (imported)	20000	1.45%	95.95%	C	
4	Water	12500	0.90%	96.86%	C	
13	Goat Cheese	12000	0.87%	97.72%	C	
12	Pepper	10800	0.78%	98.50%	C	
10	Printed Polybag	9600	0.69%	99.20%	C	Last 6 items (or 50% of the items) contribute to 5% of total inventory usage value.
8	Eggs	6000	0.43%	99.63%	C	
15	Cooking soda	4500	0.33%	99.96%	C	
6	Salt	600	0.04%	100.00%	C	

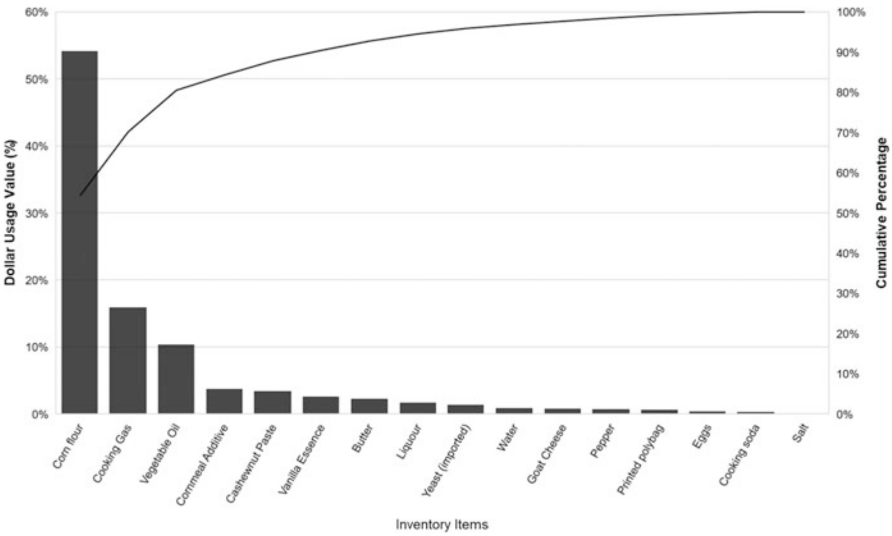

Fig. 8.1 ABC analysis

stricter control over expensive and high-usage (fast-moving) items, rather than all items equally. However, this method does have some limitations:

- The purchase price of the item can fluctuate (for e.g., gasoline), and if this happens often the ABC analysis will be required to be updated frequently making the system quite nervous.

Table 8.5 Sales data for Problem 8.1

Item	Annual sales	Profit per unit
HB2 pencil (pack)	360	$0.15
Hero fountain pen	180	$0.75
175 GSM paper (ream)	3500	$1.50
300 GSM paper (ream)	500	$2.25
Reynolds ball pen	10,000	$0.05
Pencil sharpener – Wall-mounted	300	$1.50
Sketch pen (pack)	25,000	$0.75
Stapler (micro)	2500	$2.50
Stapler (large)	100	$5.00
Stapler pins	50,000	$0.05
Painting brush – Flat	500	$8.00
Geometry set	10,000	$6.50

Table 8.6 Computing annual usage value for Solved Problem 8.1

Item	Annual sales	Profit per unit	Annual sales value
HB2 pencil (pack)	360	$0.15	$54.00
Hero fountain pen	180	$0.75	$135.00
175 GSM paper (ream)	3500	$1.50	$5250.00
300 GSM paper (ream)	500	$2.25	$1125.00
Reynolds ball pen	10,000	$0.05	$500.00
Pencil sharpener – Wall-mounted	300	$1.50	$450.00
Sketch pen (pack)	25,000	$0.75	$18,750.00
Stapler (micro)	2500	$2.50	$6250.00
Stapler (large)	100	$5.00	$500.00
Stapler pins	50,000	$0.05	$2500.00
Painting brush – Flat	500	$8.00	$4000.00
Geometry set	10,000	$6.50	$65,000.00

- An item held in stock may not be very expensive but may be very critical to the functioning of the organization operations. However, the ABC analysis does not give any importance to the criticality. It only considers the annual usage value.

  Later in this chapter we discuss other methods of classification.

Solved Problem 8.1

Table 8.5 lists 10 items sold at a retail shop. Classify each of these items into A, B, or C categories.

Solution

Let us first compute the annual usage value for each of the items. This can be obtained for each item by multiplying the profit per unit of the item with its annual sales. This is shown in Table 8.6.

Table 8.7 Rank-ordering (by Annual usage value) for Solved Problem 8.1

Item	Annual sales	Profit per unit	Annual sales value	Proportion	Cumulative proportion	Category
Geometry set	10,000	$6.50	65,000	0.6219	0.6219	A
Sketch pen (pack)	25,000	$0.75	18,750	0.1794	0.8013	A
Stapler (micro)	2500	$2.50	6250	0.0598	0.8611	B
175 GSM paper (ream)	3500	$1.50	5250	0.0502	0.9114	B
Painting brush – Flat	500	$8.00	4000	0.0383	0.9496	B
Stapler pins	50,000	$0.05	2500	0.0239	0.9736	C
300 GSM paper (ream)	500	$2.25	1125	0.0108	0.9843	C
Reynolds ball pen	10,000	$0.05	500	0.0048	0.9891	C
Stapler (large)	100	$5.00	500	0.0048	0.9939	C
Pencil sharpener – Wall-mounted	300	$1.50	450	0.0043	0.9982	C
Hero fountain pen	180	$0.75	135	0.0013	0.9995	C
HB2 pencil (pack)	360	$0.15	54	0.0005	0.0005	C
		Total	104,514			

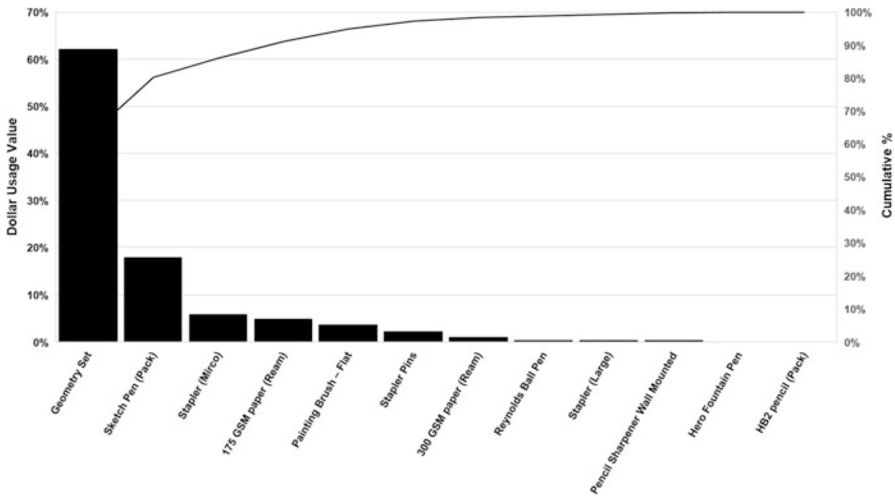

Fig. 8.2 ABC chart for Solved Problem 8.1

Next, we rank-order the items based on their annual usage value (descending order). Table 8.7 shows the rank-ordered items. The top two items (24% of items) can be categorized as A class, the next three items (36% of items) as B class, and the remaining seven items as C class. Figure 8.2 shows the ABC chart for this problem.

8.3 Exchange Curves

Exchange curves can be developed by inventory managers when cost data are unreliable or unavailable (Silver et al. 1998; Nahmias 2005). These curves can be used for easier and efficient management of multi-item inventory systems. An exchange curve is a hyperbola that shows the relationship between the value of items held in inventory and the number of replenishments per year. Consider a multi-item deterministic system that has n items. If the EOQ concept is used to manage orders for each of these items, then

$$Q_j = \sqrt{\frac{2D_jC_o}{iC_j}} \tag{8.1}$$

where D_j and C_j are the annual demand and the purchase price for the jth item held in multi-item inventory system. The total number of replenishments, N, to be placed would be

$$N = \sum_{j=1}^{n} \frac{D_j}{Q_j} \tag{8.2}$$

The total value of items, TV, held in the inventory would be

$$\text{TV} = \sum_{j=1}^{n} \frac{C_jQ_j}{2} \tag{8.3}$$

It should be noticed that in Eq. 8.1 the demand and purchase price are different while it is fairly reasonable to assume that the ordering cost and interest rate are the same for each item. Substituting the value of Q_j from Eq. 8.1 in Eq. 8.3, we can rewrite the total value of items held in inventory as

$$\text{TV} = \sum_{j=1}^{n} \frac{C_j}{2}\sqrt{\frac{2D_jC_o}{iC_j}} \tag{8.4}$$

Simplifying Eq. 8.4, we get

$$\text{TV} = \sqrt{\frac{C_o}{i}}\frac{1}{\sqrt{2}}\sum_{j=1}^{n} \sqrt{D_jC_j} \tag{8.5}$$

Similarly, we can state the number of replenishments as

$$N = \sqrt{\frac{i}{C_o}}\frac{1}{\sqrt{2}}\sum_{j=1}^{n} \sqrt{D_jC_j} \tag{8.6}$$

Table 8.8 Exchange curve – Sample calculation

Item (j)	C_j	D_j	$^{C_o}/_{i}$	Q	Orders per year	Total value of items
Corn flour	30	25,000	266.67	666.7	38	10,000
Cooking gas	4600	48	266.67	2.4	20	5426
Vegetable oil	20	7200	266.67	438.2	16	4382
Cornmeal additive	70	750	266.67	75.6	10	2646
Cashew nut paste	400	120	266.67	12.6	9	2530
Vanilla essence	300	120	266.67	14.6	8	2191
Butter	65	500	266.67	64.1	8	2082
Liqueur	200	120	266.67	17.9	7	1789
Yeast (imported)	200	100	266.67	16.3	6	1633
Water	0.5	25,000	266.67	5164.0	5	1291
Goat cheese	500	24	266.67	5.1	5	1265
Pepper	300	36	266.67	8.0	5	1200
Printed polybag	8	1200	266.67	282.8	4	1131
Eggs	25	240	266.67	71.6	3	894
Cooking soda	45	100	266.67	34.4	3	775
Salt	6	100	266.67	94.3	1	283
				Total	148	39,517

Multiplying Eq. 8.5 with Eq. 8.6, we get

$$\mathrm{TV} \times N = \frac{1}{2}\left(\sum_{j=1}^{n} D_j C_j\right)^2 \tag{8.7}$$

Equation 8.7 represents the hyperbolic curve. Dividing Eq. 8.5 with Eq. 8.6, we get the representation of any point on this hyperbolic curve given by

$$\frac{\mathrm{TV}}{N} = \frac{C_o}{i} \tag{8.8}$$

If we use the ratio of the ordering cost to interest rate as a policy parameter, we can generate an exchange curve that shows the relationship between the number of orders and the total value of items. Table 8.8 illustrates a sample calculation for an order cost to interest rate ratio of 266.67 ($C_o = \$80$ and $i = 30\,\%$).

The total number of orders in this case works out to 148, while the total value of items held in inventory is $39,517. Inventory managers can obtain an exchange curve by computing the number of orders and total inventory value for different values of $\frac{C_o}{i}$, as shown in Fig. 8.3.

Let us now apply the concept of exchange curve to Rosetta's case. We know that Rosetta's is currently operating with an ordering cost of $80 per order and an inventory carrying rate of 30% per year. The ratio of ordering cost to inventory carrying rate is 266.67. For this ratio, the number of orders is 148 and the total

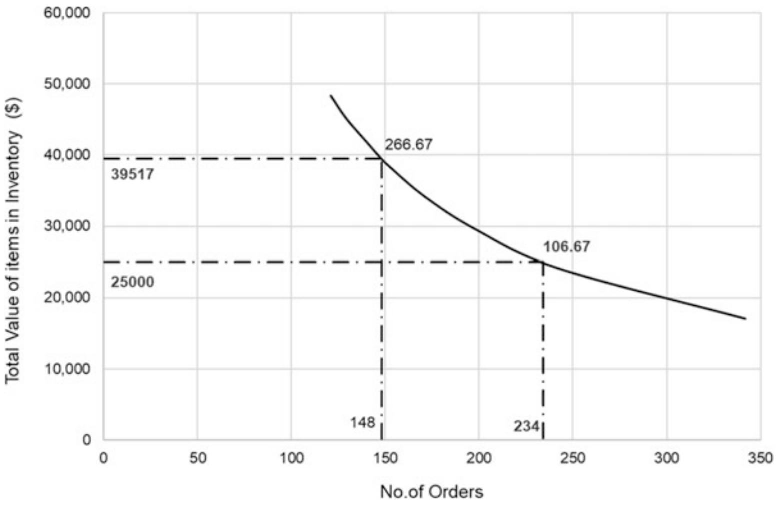

Fig. 8.3 Exchange curve for items at Rosetta's

inventory value is $39,517, as shown in Table 8.5. Let us suppose that Rosetta's is not comfortable with the total value of inventory and wants to reduce it to, say, $25,000. Rosetta's can accomplish this by reducing the ratio of $\frac{C_o}{i}$ to 106.67 (ordering cost is $32, and inventory carrying rate is 30%). From the exchange curve (Fig. 8.3), we can notice that for this ratio the total value of inventory reduces to around $25,000, while the number of orders increases to 234. In other words, Rosetta's is able to save 37% of the costs but will need to increase the number of orders from 148 to 234. Rosetta's will now have to decide if the cost savings of $14,517 is worth the effort required to place additional 86 orders. This way Rosetta's would be able to use the exchange curve and decide on a ratio they are comfortable with, as well as confirm the ordering costs and the carrying rate, which are usually difficult to estimate.

8.4 VED Classification

VED classification method categorizes items held in inventory into three classes – vital, essential, and desirable – on the basis of criticality of an item (Vrat 2014). This technique requires an inventory manager to categorize an item based on its criticality. A simple way to assign a criticality rating to an item would be to estimate the loss of production if the said item is required but not available in stock. Higher the production loss, the higher is the criticality rating of the item. Using this policy, items that are categorized vital *must* be available on hand when a need arises. Nonavailability of these items will result in significant production losses that must be avoided at any cost. Essential items are those that should be in stock when needed. Nonavailability of these items when required would result in production losses but not as significant as those like the vital category items. Desirable items are those that are good to have in the inventory and those that result in not-so-significant production losses when not available. While originally designed to manage maintenance inventories and spare parts, this technique of classification can be used to manage other types of resources as well.

Criticality of an item can be assessed on one or more factors, rather than just unavailability. Chitale and Gupta (2014) suggest that inventory managers may use a set of risk factors, including unavailability, that form the basis of estimating an item's criticality. Examples of these risk factors may include the following:

- Procurement lead time: Items with shorter lead times would be less critical while those with longer lead times would be more critical.
- Supplier: Items that are supplied locally would be less critical while those supplied by international sources would be more critical.
- Amount of customization: An item that is highly customized would be considered more critical than one that is highly standardized.
- Unavailability: An item that may not be available when needed may result in loss of production, and hence loss of revenue.

Tables 8.9 and 8.10 show the risk factors and the categorization criteria that form the basis for estimating an item's criticality.

An organization using VED to manage its inventory would have a larger stock of V class items and a relatively smaller stock of D class items. Let us consider a sample of the items from Rosetta's as shown in Table 8.11.

Let us now use the VED technique to classify these sample items into V, E, and D using information available. Consider the first item – vegetable oil. The lead time for this item is 3 days, the supplier is based locally, and there is no product customization (i.e., the product is purchased as is off the shelf). Each of these factors would be rated low (1 point). However, if the item is unavailable, it would

Table 8.9 Risk factors for VED analysis

Risk factor	High (3 points)	Medium (2 points)	Low (1 point)
Procurement lead time	More than 5 weeks	1–5 weeks	Less than 1 week
Supplier location	International	National	Local
Customization	High degree of customization	Minor customization	Highly standardized item or no customization
Unavailability	High production losses if unavailable	Medium production loss if unavailable	Low or no production loss if unavailable

Table 8.10 VED categorization criteria

Category	Score
Vital	11 and above
Essential	7–10
Desirable	4–6

Table 8.11 Sample of items for VED categorization

Item	Procurement lead time	Supplier location	Degree of customization	Unavailability
Vegetable oil	3 days	Sourced locally from Walmart	None, off the shelf	Production would need to be stopped until oil is made available
Yeast	2 weeks	International	None, off the shelf	Can produce other products as per demand
Eggs	1 day	Sourced locally from Walmart	None, off the shelf	Can produce other products as per demand
Corn flour	2 days	Sourced locally from Walmart	Minor degree of changes to the grain structure.	Production would need to be stopped until oil is made available

cause significant production losses and therefore we assign a high rating to unavailability risk factor (3 points). In all, this item is assigned a rating of 6 points. Looking up Table 8.10, we can see that items with a score of 4–6 are categorized as desirable. Table 8.12 shows the categorization plans for the sample of items.

8.5 FSN Analysis

This technique of selective inventory control classifies materials into three categories – fast-moving, slow-moving, and nonmoving – based on their consumption pattern or rate of movement. Items that are regularly consumed are classified as fast-moving, while those items that have never been consumed even once but are still maintained in the inventory are classified as nonmoving. Rest of the items are classified as slow-moving. This technique has been more popular in the management of spare parts and maintenance resources.

8.6 Other Selective Inventory Control Techniques

Several models exist in literature that combine two or more of these selective inventory control techniques for better classification and management of inventory items. Inventory managers have used, for example, a combination of ABC and VED to have tighter control. Using these two techniques together, inventory managers would be able to create nine groups of items including A–V class of items (items that have high usage value and are also vital for production) and the C–D class of items (items that have low usage value and are of type-desirable). This is shown in Fig. 8.4. Interested readers may review works of Chitale and Gupta (2014), Gopalakrishnan (2001), and Vrat (2014) for more information on these models.

Table 8.12 VED Analysis for a set of items from Rosettas

Item	Procurement lead time	Supplier location	Degree of customization	Unavailability	Total score	Category
Vegetable oil	3 days Low = 1	Local supplier Low = 1	Standardized Low = 1	High losses Medium = 3	6	Desirable
Yeast	2 weeks Medium = 2	International High = 3	Standardized Low = 1	Low losses Low = 1	7	Essential
Eggs	1 day Low = 1	Local supplier Low = 1	Standardized Low = 1	Low losses Low = 1	4	Desirable
Corn flour	2 days Low = 1	Local supplier Low = 1	Minor changes Medium = 2	High losses High = 3	7	Essential

		Usage Value of items		
		A Class	B Class	C Class
Availability of items	Vital	A-V	B-V	C-V
	Essential	A-E	B-E	C-E
	Desirable	A-D	B-D	C-D

Fig. 8.4 ABC × VED technique

8.7 Summary

Organizations hold several thousands of items in their inventory. If managers in these organizations use single item models, then all items, irrespective of their usage or criticality, would need to be treated the same way. This would not be a very efficient. In this chapter, we reviewed selective inventory control methods to manage inventory, those that require managers to focus their effort more on items that are key to the organization, and applying lesser control on other items. We reviewed popular selective inventory control methods such as ABC analysis, VED analysis, and FSN analysis. Each of these methods classifies inventory items into three groups based on item attributes. A hybrid method – combining the ABC and the VED techniques – was also discussed. In addition to these techniques we also reviewed an important multi-item inventory management technique that has EOQs as its basis – the exchange curve. We discussed how the exchange curves can help inventory managers identify an appropriate ratio of ordering cost to inventory carrying rate that can achieve a management-desired level (constraint) of inventory value.

8.8 Case Study: Exchange Curves for Multi-item Management

Fernando has a retail shop in the outskirts of Guadalajara, where he sells five different type of grains – maize, white rice, brown rice, wheat, and horse gram. Ten years ago he dropped out of school and started assisting his father run the shop. He took total ownership when his father fell sick. He now makes all inventory decisions – when to place and order, how big must the replenishment be, etc. – exactly like his father.

One fine day Fernando has a visitor to his shop. Manuel, his schoolmate, now a final year undergraduate student in Industrial Engineering at the local university, visits him to enquire his well-being. During their meeting, Manuel tells Fernando how excited he is about inventory management concepts he has learnt at the university, and he is looking at executing a real-life project as part of his undergraduate studies.

"You can use any data from my retail shop. What do you need to execute this project of yours?", asks Fernando.

"All I need is the demand data and the unit price," says Manuel "and yes, it would help if you can also supply me your carrying and ordering costs," adds Manuel.

Fernando tells Manuel that he has analyzed the demand pattern for the grains he sells, for the last 3 years, and he is convinced that the demand is steady and constant all through the year. Maize is his fastest selling item. He sells 4500 kg of it every year. The two different varieties of rice – white rice and brown rice – sell 1500 kg and 1000 kg annually. Wheat and horse gram are his slower selling items – he sells 750 kg of wheat and 250 kg of horse gram each year.

"I am sure I have been able to optimize inventory policy for each individual items," says Fernando. "I make the decision on how much to order and when based on what I have on stock. I order the same amount every time I place a procurement order," he adds.

"I will let you know exactly how much and when, when we meet tomorrow," says Fernando to Manuel, as they agree to meet once again the next day to decide on the way forward.

The next day Fernando hands over a note to Manuel which has the unit price and the order size for each of the items sold by him.

Item code	Grain	Unit price ($)	Order size (kg)
A	White rice	20	150
B	Horse gram	100	25
C	Wheat	8	100
D	Maize	22	500
E	Brown rice	75	75

Case Study Questions

(a) Create a spreadsheet solution for Fernando that indicates the current point of operations.
(b) Determine the optimal order quantity and number of orders.

Solution

We will use MS Excel to solve the problem. Create a matrix using data supplied by the retailer. This is as shown in Table 8.13. It has eight columns – item name, annual demand, item cost, and current order quantity.

Step 1: Add four new columns to the table, and name those columns as follows – total value, number of orders, implied ratio of order to carrying rate. Compute the total value of inventory for each of the items. This can be determined using Eq. 8.3. For example, consider item A. The total value for this item is

Table 8.13 Compute implied ratio

Item	Annual demand	Item cost (C)	Current order quantity (Q)	Total value (TV)	Number of orders (N)	Implied $^{C_o}/_i$	$\sqrt{D_j C_j}$
White rice (A)	1500	20	150	1500	10	150	\$173.21
Horse gram (B)	250	100	25	1250	10	125	\$158.11
Wheat (C)	750	8	100	400	7.5	53	\$77.46
Maize (D)	4500	22	500	5500	9	611	\$314.64
Brown rice (E)	1000	75	75	2813	13.33	211	\$273.86
				11,463	49.83		\$705.00

$$\frac{QC}{2} = \frac{150 \times 20}{2} = 1500$$

Also, compute the sum of the total value of the five items. As shown in Table 8.13, the sum of the total value of items in inventory system is \$11462.5

Step 2: Determine the number of orders. We may use Eq. 8.2 to obtain the number of orders for each of the items. For example, the demand for item A is 1500 units and its current order quantity is 150. The number of orders, therefore, is

$$\frac{D}{Q} = \frac{1500}{150} = 10$$

Step 3: Compute the implied ratio of ordering cost to the carrying interest rate. This can be computed using the standard EOQ formula. Also, compute $\sqrt{D_j C_j}$ for each item.

Step 4: Compute new EOQ values for each of the items using different values of implied ratio. This computation is shown in Table 8.14 under columns EOQs. Equation 8.6 can be used to compute N while Eq. 8.5 can be used to compute TV.

Step 5: Recalculate N for each of the items, using the new EOQ values. This is shown under columns suffixed N (last five columns).

Copy contents (including formulae) from two rows and paste it as shown in Fig. 8.5. These two rows would be used for the GOAL SEEK function (See Strakos 2016). Use GOAL SEEK on the first row to find N for the current value of TV (\$11,463, see Table 8.15) and by varying the implied ratio (see Fig. 8.5).

Using the second row, you can find the value of TV by using the new value of N and varying the value of implied ratio. Results of GOAL SEEK are shown in Table 8.15. The exchange curve with the current operating point is shown in Fig. 8.6.

Table 8.14 Compute revised EOQs and N

A/r	N	TV	EOQ – White Rice	EOQ – Horse Gram	EOQ – Wheat	EOQ – Maize	EOQ – Brown Rice	N – White Rice	N – Horse Gram	N – Wheat	N – Maize	N – Brown Rice
50	100	$4985.10	87	16	97	143	37	17	16	8	31	27
100	71	$7050.00	122	22	137	202	52	12	11	5	22	19
150	58	$8634.45	150	27	168	248	63	10	9	4	18	16
200	50	$9970.21	173	32	194	286	73	9	8	4	16	14
250	45	$11,147.03	194	35	217	320	82	8	7	3	14	12
300	41	$12,210.96	212	39	237	350	89	7	6	3	13	11
350	38	$13,189.34	229	42	256	378	97	7	6	3	12	10
400	35	$14,100.00	245	45	274	405	103	6	6	3	11	10
450	33	$14,955.31	260	47	290	429	110	6	5	3	10	9
500	32	$15,764.28	274	50	306	452	115	5	5	2	10	9
550	30	$16,533.72	287	52	321	474	121	5	5	2	9	8
600	29	$17,268.90	300	55	335	495	126	5	5	2	9	8
650	28	$17,974.04	312	57	349	516	132	5	4	2	9	8
700	27	$18,652.55	324	59	362	535	137	5	4	2	8	7

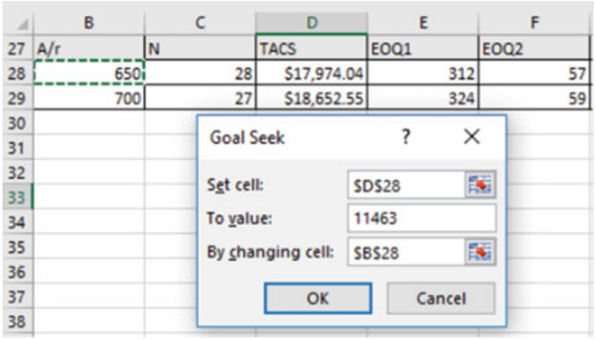

Fig. 8.5 Using GOAL SEEK in MS Excel

8.9 Practice Problems

Problem 8.1
Table 8.16 shows data for 16 items managed by an inventory manager. Perform ABC analysis to categorize these items into A, B, and C classes.

Answer:

Class	Number of items	Items	% Inventory value
A	3	Items 5, 15, and 1	78.5
B	5	Items 10, 2, 9, 3, and 11	16.2
C	8	Items 16, 8, 4, 7, 6, 13, 14, and 12	5.3

Problem 8.2
Draw an exchange curve using the data for 16 items in Table 8.16.

Table 8.15 GOAL SEEK to determine optimal N and TV

A/r	N	TV	EOQ – White Rice	EOQ – Horse Gram	EOQ – Wheat	EOQ – Maize	EOQ – Brown Rice	N – White Rice	N – Horse Gram	N – Wheat	N – Maize	N – Brown Rice
650	28	$17,974.04	312	57	349	516	132	5	4	2	9	8
700	27	$18,652.55	324	59	362	535	137	5	4	2	8	7

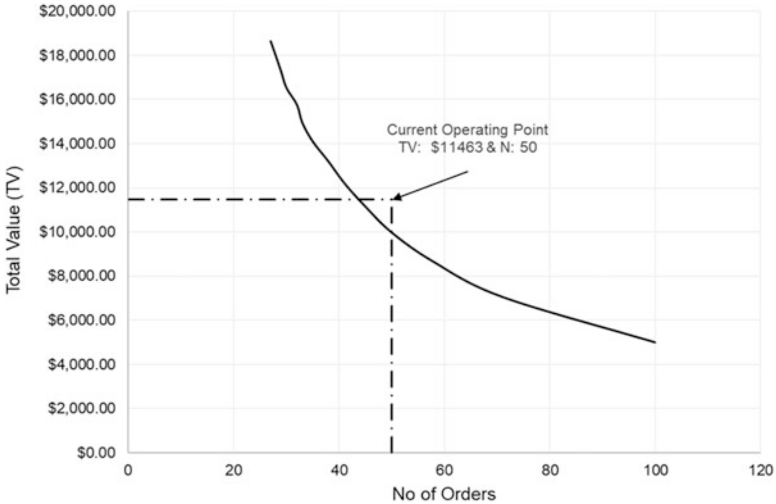

Fig. 8.6 Exchange curve for Case Study Problem

Table 8.16 Data for practice problem

Item name	Price ($)	Annual demand	Item name	Price ($)	Annual demand
Item 1	3.5	30,000	Item 9	2.5	6500
Item 2	12	1500	Item 10	15	1250
Item 3	3.5	4500	Item 11	5	2500
Item 4	15	200	Item 12	2	30
Item 5	6.5	24,000	Item 13	8	12
Item 6	4	500	Item 14	5.5	12
Item 7	0.2	12,000	Item 15	11	12,000
Item 8	7.5	1100	Item 16	1.5	7000

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Part IV
Advanced Inventory Models

Chapter 9
Inventory Models for Perishable Items and Style Goods

9.1 Introduction

Corn tortillas are sold to retail customers through their only sales outlet, managed by Maria Fernanda. Tortillas are sold in packs of 10. Each pack of 10 costs $10, and they sell it to customers at $25. Because they do not add any preservatives, the shelf life of a pack of tortilla is 1 day (24 h). Maria accepts a predetermined number of corn-flavored tortilla packs each morning at 7:00 am when the sales outlet opens. At 11:00 pm when outlet closes, she discards all the unsold packs. She also informs the kitchen manager the number of packs she would need the next morning. One of her key performance indicator (KPI) is to minimize the losses due to unsold inventory, and for this she relies on historical data. Maria must know how many packets she would need at the beginning of the day, each day.

Maria does not like wasting food. Nobody does. She has approached a food processing unit in León that reprocesses food items. The reprocessing unit has agreed to buy unsold packets of tortillas from Rosetta's at $5 per packet. Maria would now have to analyze the new situation and decide on the number of packets she would need each morning when the outlet opens for sale.

In this chapter, we determine order quantity of items that have a finite shelf life.

Perishable items and style goods inventory are important classes of inventory problem. In all of the models discussed earlier in this book we assumed that the item being procured had an infinite lifetime, and is sold over indefinite periods of time. However, in reality, at some point in its lifetime, the utility value of items drops to

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Table 9.1 Lifetime of some items

Item	Probable lifetime
Newspaper	1 day
Pastry produced in a bakery	2 days
Smartphone	18 months
Vegetables	2–5 days
Blood stored in a blood bank	21 days
Handbag for ladies (designer)	One season
Seasonal clothing (such as rain coats, sweaters, etc.)	One season
Operating system or a computer	Till the next version is released

zero (Nahmias 2005). There are no takers for those items thereafter. Examples of this class of inventory items include fashion products, food products such as milk and processed meat, bottles of whole blood, airline tickets for a specific flight, tickets for a specific soccer match, a daily newspaper, etc. Table 9.1 shows probable lifetime of some of the items.

There are other models that are based on the utility value of an item (or a group of items) during its lifetime. These include:

(a) Decaying inventory items
(b) Obsolete inventory items

The following is a brief description of the differences between the perishable inventory items being discussed in this chapter, and others that belong to a similar category.

Decaying Inventory Items
In this category of inventory problems, a small percentage of items held in the inventory system is lost due to damage or decay. Decay can be at a constant rate or at an exponential rate. An example of this would include the produce that a local vegetable vendor procures from a wholesaler. During transportation, or in storage, a small amount of the procured items may get damaged and would not be saleable.

Obsolete Inventory Items
An example of this would be an electronic item (such as a new version of a tablet computer) that has just arrived in the market. The value of such items is usually big at the beginning of the selling period but reduces rapidly once an enhanced version of the item is about to be introduced in the market.

Depending on their lifetime, perishable items can be categorized into two – those with fixed lifetime and those with stochastic (random) lifetime (Fig. 9.1). Items with fixed lifetime have a constant utility value over a fixed period of time.

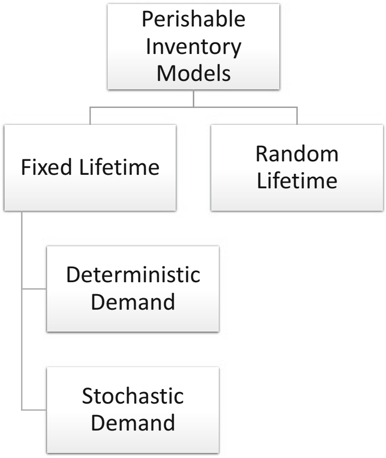

Fig. 9.1 Perishable items – Classification

Examples of this would be a daily newspaper or a unit of blood stored in a blood bank. Items with random lifetime are those whose lifetime reduces throughout their lifetime. Examples of this would include drugs (medicines), fruits, vegetables, etc. The scope of this chapter is the first category of items where the utility value remains constant throughout their lifetime and drops to zero at some point in time.

Mathematical complexity involved in modeling this class of problems is very high. Therefore, we will discuss only basic inventory models in this class. Interested readers may refer to the work by Nahmias (2011) for a complete review of models involving items with random lifetime.

9.2 Perishable Items: Deterministic Demand

If the demand is deterministic, the EOQ formula (Chap. 3) may be used to determine the optimal order quantity. The cycle time between the placement of orders is

$$T = \frac{Q}{D}$$

Nahmias (2011) has argued that if the perishable item under consideration has a useful lifetime of t, then there are two cases that need to be analyzed:

(a) $T \leq t$, and
(b) $T > t$

In the first case, all the units would be consumed by the demand in time T before the item perishes. However, small modification would need to be made for the second case since at the end of the cycle time there would be a few unsold units that would perish. The quantity to be ordered, Q, would need to be amended to

$$Q = Dt$$

9.3 Single Period Inventory Model with Stochastic Demand

A single period inventory model is one where *one* order of an appropriate size is placed for an item at the beginning of the period. At the end of the period, the stock of items is either completely sold off or any remaining items would need to be disposed off, some of which may have a salvage value. A single period inventory model, therefore, is apt to manage perishable items and style goods (Silver et al. 1998; Anderson et al. 2016). The decision variable in single period inventory model is the size of the order.

9.3.1 Assumptions

Following assumptions have been made to model inventory management of perishable items and style goods:

- The model is meant for a single item that has a finite lifetime.
- The item cannot be sold after its life is complete. However, in some circumstances, there is possibility of salvaging some value from the expiring item.
- Beginning inventory is zero. Toward the end of this chapter, we also discuss solutions involving a positive beginning inventory.
- While the demand is a random variable and is not known with certainty, the probability distribution of the demand is known.
- Shortages are not back-ordered.
- No ordering cost is incurred.

9.3.2 Relevant Costs

The objective of the single period inventory model is to balance two costs – cost of underage and cost of overage – which are defined as follows (Anderson et al. 2016):

- Cost of underage, C_u, is the cost per unit of underestimating demand. This cost represents the opportunity loss of not ordering one additional unit and finding that it could have been sold.
- Cost of overage, C_v, is the cost per unit of overestimating demand. This cost represents the loss of ordering one additional unit and finding that it cannot be sold.

Let us assume that Q units have been purchased in a given period. If p is the probability of selling Q units, then $(1 - p)$ is the probability of selling less than Q units. If the demand, D, is greater than Q units, then underage cost is incurred, that is, an additional unit (or additional units) could have been sold. The expected value of this loss is given as

$$= C_u \times p_{(D>Q)} \tag{9.1}$$

If the demand, D, is less than Q units, then an overage cost is incurred. The expected value of this cost is given as

$$= C_v \times p_{(D \leq Q)} \tag{9.2}$$

Because there are only two possibilities, either $D \leq Q$ or $D > Q$, the sum of their probabilities is 1. In other words,

$$p_{(D>Q)} + p_{(D \leq Q)} = 1$$

or

$$p_{(D>Q)} = 1 - p_{(D \leq Q)} \tag{9.3}$$

Using Eq. 9.3, we can rewrite Eq. 9.1 as

$$= C_u \times \left(1 - p_{(D \leq Q)}\right) \tag{9.4}$$

The optimal number of units, Q, we would like to buy is the one where

$$C_v \times p_{(D \leq Q)} = C_u \times \left(1 - p_{(D \leq Q)}\right) \tag{9.5}$$

Solving for $p_{(D \leq Q)}$, we get

$$p_{(D \leq Q)} = \frac{C_u}{C_u + C_v} \tag{9.6}$$

Eq. 9.6 is also known as the critical ratio, or C_r, which can be restated as

$$C_r = \frac{C_u}{C_u + C_v} \tag{9.6a}$$

Consider a fixed life item that costs \$100 and sells at \$150. Every unit sold makes a profit of \$50. Let us assume we underordered and there is a lost sale. Cost of underage is the lost profit, that is, the difference between the revenue and the cost, which in this case is $\$150 - \$100 = \$50$. Let us further assume that unsold items can be sold in a clearance sale and we can salvage \$80 for the item. The cost of overage would be increase in profit we would have enjoyed had we ordered one unit less, which is the difference between the cost and the salvage value, in this case \$100 – \$80 = \$20. Using Eq. 9.6a, we get (Sheffi 2010)

$$C_r = \frac{\text{Revenue} - \text{Cost}}{(\text{Revenue} - \text{Cost}) + (\text{Cost} - \text{Salvage})} \tag{9.7}$$

or

$$C_r = \frac{\text{Revenue} - \text{Cost}}{\text{Revenue} - \text{Salvage}} \tag{9.7a}$$

Let us now consider a slightly different situation – that of a maintenance store. A maintenance store stocks spare parts for equipment used in production processes. The primary objective of a maintenance store is to minimize costs, including production losses caused due to nonavailability of a spare part. In the event of a stockout, the store manager must order a spare from the original equipment manufacturer (OEM). When production losses are significant, the store manager may have to get the spare airlifted. The cost of doing this, also referred to as cost of resupply, would be very high. A summary of all the costs involved is as follows:

- Cost of a spare
- Cost of holding an unused spare in inventory
- Cost of resupplying a required spare part

The critical ratio in this situation can be stated as

$$C_r = \frac{\text{Resupply Cost} - \text{Cost of spare}}{\text{Resupply Cost} + \text{Cost of holding unused spare}} \tag{9.8}$$

Depending on the situation, either of Eq. 9.6a, Eq. 9.7a, or Eq. 9.8 may be used to determine the critical ratio. In the following sections, we discuss application of the critical ratio concept to a variety of problems involving known probability distributions.

A Note on Style Goods

Style goods are also characterized by short (duration) selling period and limited replenishment opportunities and salvage. In some sort, style goods and fashion products may also be treated as perishable items. In this book, we use the single period inventory model to solve style goods problem as well.

9.3.3 Case of Normally Distributed Demand

Let us go back to our running example of Rosetta's. The front office manager at Rosetta's has to place an order for a certain number of tortilla packets each day for sale through their retail outlets spread across the city of Leon. Each packet of tortilla costs her \$10 and she retails it at \$25 per packet. Weekly demand for tortilla is as shown in Table 9.2.

How many packets of tortillaS must the front desk manager order if:

- The unsold packets of tortilla at the end of the day's business perishes, and has no salvage value?
- The unsold packets are sold to a nearby food processing firm at \$5 per packet?

To start the solution process, we assume that the demand is normally distributed. (Later in this chapter we will discuss solutions for items whose demand follows a uniform or Poisson distribution as well.) Note that we do not have parameters of the normal distribution. The first step, therefore, is to compute the parameters of the normal distribution – the mean and standard deviation. Table 9.3 illustrates computation of these parameters.

The mean weekly demand, μ, is

$$= \frac{\sum fX}{\sum f} = \frac{99588}{176} = 565$$

Next we can calculate the standard deviation, σ, which is given by

$$= \sqrt{\frac{\sum f\left(X - \bar{X}\right)^2}{\sum f}} = \sqrt{\frac{7388580}{176}} = 205$$

Thus, the mean demand is 565 and standard deviation is 205 packets. The next step is to compute the critical ratio. We know that the revenue per packet is \$25 and the cost per packet is \$10. Moreover, unsold tortilla packets have no salvage. Substituting these values in Eq. 9.7a, we get

Table 9.2 Weekly demand for tortillas at Rosetta's

Lower bound	Upper bound	Frequency	Lower bound	Upper bound	Frequency
0	100	2	501	600	30
101	200	7	601	700	35
201	300	12	701	800	28
301	400	18	801	900	15
401	500	23	901	1000	6

Table 9.3 Calculation of Mean and Standard Deviation – Demand for Tortillas at Rosetta's

Lower bound	Upper bound	Frequency (f)	Class midpoint (X)	fX	$X-\bar{X}$	$(X-\bar{X})^2$	$f(X-\bar{X})^2$
1	100	2	50.5	100	−515.8	266086.0	532172.0
101	200	7	150.5	1053.5	−415.3	172503.4	1207523.5
201	300	12	250.5	3006	−315.3	99436.3	1193235.7
301	400	18	350.5	6309	−215.3	46369.3	834646.7
401	500	23	450.5	10361.5	−115.3	13302.2	305950.9
501	600	30	550.5	16,515	−15.3	235.2	7055.1
601	700	35	650.5	22767.5	84.7	7168.1	250884.3
701	800	28	750.5	21,014	184.7	34101.1	954830.2
801	900	15	850.5	12757.5	284.7	81034.0	1215510.5
901	1000	6	950.5	5703	384.7	147967.0	887801.9
		$\sum f = 176$		$\sum fX = 99588$			$\sum f(X-\bar{X})^2 = 7{,}388{,}580$

$$C_r = \frac{25 - 10}{25 - 0} = 0.60$$

Since we have assumed the demand follows a normal distribution, we need to find the value of z, the standard normal variate, for a probability of 0.60. We can use MS Excel function, NORM . S . INV(C_r), to obtain this:

$$z = \text{NORM.S.INV}(0.60) = 0.2533$$

The optimal order quantity Q is given by

$$Q = \mu + z\sigma \tag{9.9}$$

We know the mean is 565 packets and the standard deviation is 205 packets. Substituting the values in Eq. 9.9, we get

$$Q = 565 + (0.2533 \times 205) = 618 \text{ units}$$

The quantity to be ordered is 618 packets.

Let us now consider the scenario where unsold packets can be salvaged at $5 per packet. The critical ratio in this case is

$$C_r = \frac{25 - 10}{25 - 5} = 0.75$$

Using the MS Excel function NORM . S . INV(p) to obtain the standard normal variate, we get

$$z = \text{NORM.S.INV}(0.75) = 0.675$$

Substituting the values of μ , σ, and z in Eq. 9.9, we get

$$Q = 565 + (0.675 \times 205) = 704 \text{ units}$$

The front office manager at Rosetta's has to place an order for 704 tortillas each day for sale through their retail outlet.

Solved Problem 9.1

H&N is a high-end fashion goods retailer. They source their goods from leading brands and sell those through their retail shops. CoolColors, a leading informal shirt brand has just given H&N a preview of their new designer after-hours shirt for men. H&N would like to buy the designer shirts and sell those through their retail outlets. Each shirt costs $35. H&N believes they would be able to sell each shirt at $55 during the winter (November–February). Any unsold shirt can be sold during the clearance sale in early March for $30 each. Compute the optimal order quantity, assuming the demand for the shirt is normally distributed with a mean of 200 and a standard deviation of 25.

Solution

We start the solution process by computing the critical ratio. We know the cost is \$35 and the sale price is \$55. The salvage value is \$30. Using Eq. 9.7a, we get

$$C_r = \frac{55 - 35}{55 - 30} = 0.80$$

The next step is to find the value of the standard normal variate for a probability of 0.80. We can use MS Excel function, NORM . S . INV(p), to obtain this:

$$z = \text{NORM.S.INV}(0.80) = 0.842$$

The optimal order quantity Q can be computed using Eq. 9.9. Substituting the values in Eq. 9.9, we get

$$Q = 200 + (0.842 \times 25) = 221 \text{ shirts}$$

The optimal order quantity is 221 shirts.

9.3.4 *Case of Uniformly Distributed Demand*

Eq. 9.9 may be used only when demand is normally distributed. However, when demand is uniformly distributed, Q can be computed using the following:

$$Q = B_L + [C_r \times (B_U - B_L)] \tag{9.10}$$

where

B_L is the lower bound;
B_U is the upper bound; and
C_r is the critical ratio.

Solved Problem 9.2

Consider the H&N situation in Solved Problem 9.1. Compute the optimal order quantity assuming the demand for the shirt is uniformly distributed with a likely sale of minimum 100 and maximum 300 shirts.

Solution

We have the following data with us:

- Critical ratio $C_r = 0.8$
- Upper bound, $B_U = 300$
- Lower bound, $B_L = 100$

Substituting these values in Eq. 9.10, we get

$$Q = 100 + (0.8 \times 200) = 260 \text{ shirts}$$

Solved Problem 9.3
SpiceJet sells tickets for their Bangalore–Mumbai flight SJ516 that has a capacity of 325 seats. SpiceJet strategists have analyzed flight load data and have found that between 20 and 35 passengers cancel their tickets. SpiceJet, therefore, overbook their flights. They make a profit of $125 on every paid seat. But when the flight is full, an additional passenger arriving at the check-in counter has to be awarded free flights or cash payment, which costs SpiceJet $50. Compute the number of tickets SpiceJet may overbook for flight SJ516.

Solution
In this problem,

- The cost of underage is revenue lost due to an empty seat on the flight ($125);
- The cost of overage is the cash payment that SpiceJet needs to make to those passengers that cannot be taken on board ($50).

Using Eq. 9.6a, we have

$$C_r = \frac{125}{125 + 50} = 0.71$$

Since a minimum of 20 and a maximum of 35 passengers cancel their tickets, from Eq. 9.10, we have

$$Q = 20 + [0.71 \times (35 - 20)] = 31$$

SpiceJet may overbook 31 seats for their flight SJ516, or sell tickets for 356 seats.

9.3.5 Case of Poisson Distributed Demand

When demand follows a Poisson distribution (Jenson and Bard 2003), the following MS Excel function can be used to compute the individual and cumulative probabilities:

$$p(n) = \text{POISSON}(x, \mu, CF)$$

where

- $p(n)$ is the probability of having a demand of n items;
- x takes different values: 0, 1, 2, 3, 4. . .n items;

- μ is the sample mean;
- CF is the cumulative flag; it is set to TRUE for obtaining cumulative probability of the Poisson distribution.

For example, consider an item whose demand follows a Poisson distribution with a mean of 5 units. The probability that the demand in any period would be *exactly* 2 units is

$$p(2) = \text{Poisson}(2, 5, \text{FALSE}) = 0.08$$

In other words, there is 8% chance that the demand for the item would be exactly 2 units.

Solved Problem 9.4[1]

A generator on board a naval ship has a critical component. In the event this component fails, it needs to be replaced with a spare one held on board. The commander of the ship needs to determine the number of spares of this component he would need to hold on board at the start of their yearlong ¨at-sea¨ period. One spare component costs $15,000. Nonavailability of the spare component on board would necessitate the commander to request a helicopter to airlift a replacement from the nearest onshore depot. The cost of this exercise would be $100,000. The component takes up a large space on board the ship, and the holding costs and the cost of nonusage is approximately $50,000. The observed failure rate of this component is four failures per year.

Solution

Table 9.4 shows the probability values for Poisson distribution using a mean of 4. This can be generated using the MS Excel function POISSON(n, 4, TRUE) where n is the number of spares to be held (0, 1, 2, etc.).

We can use Eq. 9.8 to determine the critical ratio, which in this case is

$$C_r = \frac{100000 - 15000}{100000 + 50000} = 0.567$$

The critical ratio corresponds to a value between 3 and 4 in Table 9.4. It is therefore advisable to carry four spares on board before they leave for their yearlong expedition.

Table 9.4 Poisson values for Mean = 4

Number of spares	Cumulative probability
0	0.0183
1	0.0915
2	0.2381
3	0.4334
4	0.6288

[1] Adapted from Jensen and Bard (2003).

9.3.6 Case of Discrete Distribution

In the previous section we assumed, demand followed a known distribution. In some cases, organizations may not have sufficient demand data to fit it to any known distribution. In such a case we can use a graphical method to obtain optimal order quantity (See Monks 1987). Consider a situation where a campaign is being run to administer polio vaccine (“polio drops”) to children under the age of 5 in Mexico. Administration of the vaccine is done on the first Monday of every quarter in government-managed hospitals. Hospital Angeles in Leon is one such hospital where the vaccine is administered. One day before the event, the hospital management has to place an order for a certain number of vaccines, or shots, they would likely be administering. It costs the hospital management $2 to purchase one shot, and they charge $3 for administering it on a child. Unused vaccines are destroyed/disposed off since they are unusable for the next quarter. The hospital management needs to decide the quantity of vaccines that would need to be ordered. Since the hospital has been administering these shots for the last 6 years (24 quarters), they have been able to compile the demands for vaccines shown in Table 9.5

The first step is to draw a histogram to see if the data can be fit to any known distribution. Fig. 9.2 shows the histogram. Notice that the histogram reveals no pattern, and therefore it would be incorrect to apply the formulae we used in the previous section.

Table 9.5 Quarterly demand for Polio Shots

15	18	21	15	17	22	20	16
22	18	25	15	26	29	16	30
17	25	18	16	19	23	20	15

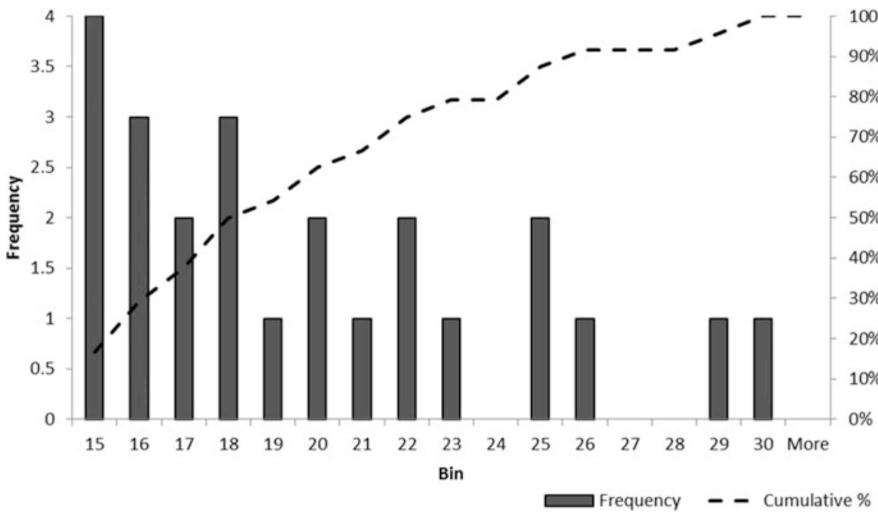

Fig. 9.2 Quarterly demand for Polio Shots – Histogram

Table 9.6 Demand for Polio Shots – Probability distribution

Number of instances	Probability	Cumulative probability	Number of instances	Probability	Cumulative probability
15	0.166667	0.166667	23	0.041667	0.791667
16	0.125	0.291667	24	0	0.791667
17	0.083333	0.375	25	0.083333	0.875
18	0.125	0.5	26	0.041667	0.916667
19	0.041667	0.541667	27	0	0.916667
20	0.083333	0.625	28	0	0.916667
21	0.041667	0.666667	29	0.041667	0.958333
22	0.083333	0.75	30	0.041667	1

The next step is to construct a probability distribution table. This is shown in Table 9.6. The probability can be computed by dividing the demand value by the number of times the demand has been observed. For example, a demand for 16 vaccines has been observed three times in the last 24 quarters. The probability of demand for 16 vaccines is $3/24 = 0.125$. We can also compute the cumulative probability values.

The next step is to calculate the critical ratio, C_r. We know that the revenue per vaccine is \$3, and the cost per vaccine is \$2. It is noted that there is no salvage value. Substituting these values in Eq. 9.7a, we get

$$C_r = \frac{3 - 2}{3 - 0} = 0.33$$

The critical ratio of 0.33 corresponds to $Q = 17$ units (from Table 9.6). The hospital management must place an order for 17 vaccines.

9.4 Graphical Approach

Consider Rosetta's running example we discussed in Sect. 3.3. We can use a graph to obtain the optimal order quantity value. A graph that represents the cumulative distribution of the demand can be drawn as shown in Fig. 9.3. We know the critical ratio is 0.60 (no salvage value case). It is noticed that the critical ratio ($C_r = 0.6$) corresponds to a value of 635 on the cumulative distribution curve. The optimal order quantity is, therefore, 635 units (compared to 618 packets using the analytical method).

Also, for the case where there is a salvage value of \$5 (critical ratio is 0.75), the optimal quantity is 712 packets (704 packets using analytical method; see Sect. 3.3).

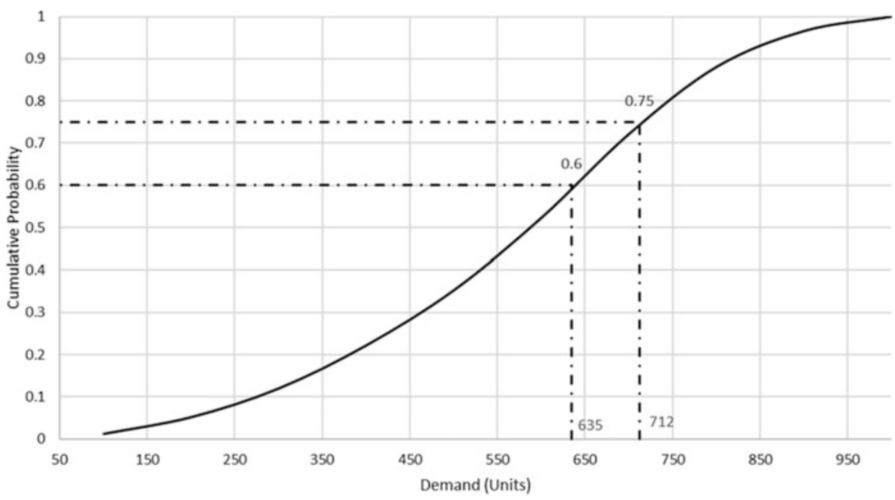

Fig. 9.3 Graphical method

9.5 Incorporating Beginning Inventory

One of the assumptions made earlier in this chapter is that the inventory at the beginning of the period is zero. This assumption is particularly valid for items that perish at the end of the period. But there may be products whose life exceeds one period. In such a case, if the beginning inventory is a positive value, then the following decision rules may be used (Nahmias 2005):

- If the beginning inventory is less than Q, then subtract the beginning inventory from the order quantity.
- If the beginning inventory is more than the Q, then do not place an order.

Solved Problem 9.5
Consider the situation in Solved Problem 9.1. The optimal order quantity is 221 shirts. If H&N already has 30 shirts on hand they received from another source, what would be the size of the order?

Solution
In this problem, since beginning inventory is less than Q, H&N must place an order for $Q - 30 = 191$ shirts.

9.6 Summary

Several items have limited shelf life. Their utility value reduces to zero at certain point in time after which these items are no longer demanded by customers. These are called perishable items. In this chapter we discussed inventory models for perishable items, also called single period inventory models or newspaper boy problems. We considered items whose demand follows a known discrete or continuous distribution (normal, uniform, Poisson). We also discussed the solution process for items whose demand pattern does not fit any known distribution. Due to similarities in the characteristics, the solution approach used for managing perishable items may also be used to solve style goods problems as demonstrated in this chapter.

9.7 Case Study: Managing Sales of Sports Gear

Soccer is the most popular team sport in Mexico, and La Liga MX is Mexico's premier soccer league. The soccer league is administered by the Mexican Soccer Association (MSA). Each year, 18 teams participate in this soccer league. The soccer season starts in March and ends in May. Each team plays the other 17 teams once, and the league table topper is crowned the champion for the year.

Armando is the proprietor of S-Mart, a very popular retail chain involved in selling sports goods all over Mexico. Armando buys sports goods directly from the source and sells it through his 35 retail shops spread across the country.

To cash in on the soccer-frenzy public, the MSA administrators introduce new designs of sports gear including footwear, jerseys, coffee mugs, and the like each year. Well before the start of the season, the league administrators invite top retailers from Mexico to a special preview session where they present new sports gear. Their objective is to obtain a commitment from the retailers and distributors on the quantity they would be able to sell through their retail shops. Being the owner of a very popular sports good retail chain, Armando regularly gets called for such special preview sessions.

This year Armando has been offered sole rights to sell jerseys that have the logos of all the 18 participating teams. Under the deal, MSA would sell Armando all the jerseys at $60 per piece. Armando is confident he would be able to retail it at $95 per piece this season. Unsold stock of jerseys would have to be put through a clearance sale. He expects to sell all of this (unsold) stock at $50 per piece. Armando has to now make a decision on the number of jerseys he would be able to sell. He has compiled data of the sale of jerseys (in thousands) over the last 30 seasons, as shown in Table 9.7

Table 9.7 Historical demand (in thousands) for La Liga MX Jerseys

61	93	69	75	58	95
87	55	76	87	79	112
71	85	63	53	68	96
88	81	85	77	83	108
91	101	92	105	94	106

Case Study Questions

(a) Compute the mean and standard deviation for the data shown in Table 9.7.
(b) Use a graphical method to estimate the number of jerseys that Armando would be able to sell.
(c) Assuming the data follows normal distribution, analytically determine the number of jerseys Armando would need to buy from MSA, Use the parameters you obtained as part of (a) above.
(d) Compare the results obtained using the graphical method and the analytical method.
(e) How different would the estimates be if Armando assumes the sales pattern follows a uniform distribution with a minimum of 80,000 jerseys and a maximum of 110,000 jerseys.

9.8 Practice Problems

Problem 9.1
All standard rooms at Hotel Gateway, located next to the cricket stadium, are fully booked during the cricket premier league season. Cancellation occurring during the season is uniformly distributed between five and eight rooms. When a customer with reservation arrives at the check-in counter and there are no rooms available, the hotel authorities must accommodate this customer at another partner hotel, but it costs them \$50. If the tariff of a standard room is \$35, determine the number of rooms that Hotel Gateway staff may overbook.

Hint
The cost of underestimating cancellation is \$35 and the cost of overestimating cancellation is \$50. Use these values to compute critical ratio.

Answer
Critical ratio is 0.41; number of rooms that may be overbooked is 6.23 (~7 rooms).

Problem 9.2
Consider the overbooking situation in Problem 9.1. If the demand is normally distributed with a mean of 5 and standard deviation of 3, determine the number of rooms that the hotel staff may overbook.

Answer:
For a critical ratio of 0.41, the NORM.S.INV() function returns a negative value (−0.28). This means that the number of rooms that may be overbooked is less than the mean:

$=5 + [(-0.2275) \times 3] = 4.3$ (~4 rooms)

Problem 9.3
Eagle Calendars is a retail shop that buys and sells year planners. Each year in November they receive one bulk order for planners. It costs Eagle \$1.5 per planner

and they sell them for \$4.5 each. In February, unsold planners are sold in bulk to a paper and pulp mill for \$0.25 each. How many calendars should Eagle order if the demand during the peak season is uniform distribution between 12000 and 15000.

Answer

$z = \text{NORM.S.INV}(0.705) = 0.53.$

Order size $= 12000 + (0.53 \times 3000) = 13590$

Problem 9.4

A newspaper boy purchases newspapers from an agency every morning at \$0.50 per copy. He gets a commission of \$0.25 for every copy sold. Unsold newspapers can be returned to the news agency at \$0.10 per copy. If the daily demand for newspapers is normally distributed with a mean of 15 and a standard deviation of 3, determine the number of newspapers that the newspaper boy must procure from the agency.

Answer:

Underage cost is the commission: \$0.25; overage cost is \$0.50 – \$0.10 = \$0.40; critical ratio = 0.385, NORM.S.INV returns: −0.25; therefore, the number of papers to be bought would be less than the mean ($15 - 0.76 \sim 14$ papers).

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Chapter 10
Inventory Models for Maintenance and Repairable Items

10.1 Introduction to Maintenance Inventories

> Rosetta's uses large automated tortilla-making machines. Each machine includes a dough loading section, a pressing section, a spiral heating (carousel) section, and a packaging section. The spiral heating section is driven by a powerful three-phase, 230 V electric motor that is used 365 days a year for 16 hours each day. In the event of a motor failure, production losses incurred by Rosetta's would be $100 per hour. The failure rate (p) of the motor is 2 per year, and the mean time to repair (T) is 3 months. Rosetta's has four such identical tortilla-making machines, and currently has one spare rotable motor in their stores.
>
> How many spare motors must Rosetta's maintain in their inventory? Is one spare motor held in inventory sufficient for Rosetta's to minimize their production losses? What would be the savings in downtime if they employ one more spare motor? These are some of the questions that would be addressed in this chapter.

Organizations use different equipment to transform raw materials into finished products. An equipment is made up of several parts – systems, subsystems, components, subcomponents, etc. – with each part having its own defined function. These parts may fail any time during the lifetime of the equipment. When this happens, the failed part needs to be replaced with another (new or as good as new) part. A replacement is also sought when a part is not functioning as expected. This replacement part is referred to as a spare part.

An equipment needs regular maintenance to ensure it delivers expected service performance. Routine maintenance activities also consume materials – for example, lubricants, coolants, cleaning chemicals, etc. Sophisticated equipment have inbuilt

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sensors to predict an impending failure of a part or subsystem. These sensors may also fail, requiring them to be replaced with new ones. All these materials – spare parts, lubricants, coolants, sensors, etc. – that support the revenue-generating processes of an organization are referred to as maintenance items. These items are stored in the maintenance stores, because their nonavailability may result in significant equipment downtime, and hence loss of business profits. These items form part of the maintenance inventories. In this book, we use the terms spare parts and maintenance items interchangeably.

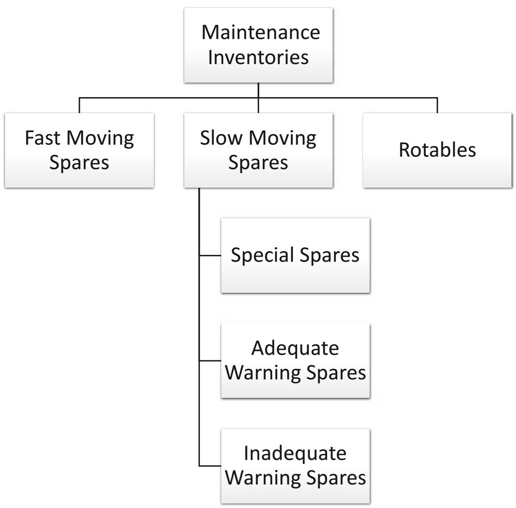
10.2 Classification of Maintenance Inventories

Based on the usage rate, Kelly (2006) has classified spare parts as follows (we use the same classification for most maintenance items):

- Fast-moving items: These are maintenance items that have a demand of 3 units or more per year.
- Slow-moving items: Maintenance items that have a demand of fewer than 3 units per year are considered to be slow-moving spare parts. Mitchell (1962) has further classified slow-moving items into specials, adequate warning, and inadequate warning spares, based on the amount of time that is available to react to an impending failure.
- Rotables: In most circumstances, failed parts are removed from an equipment and are replaced by new ones from the stores. In some cases, it is possible to repair failed parts. Rotables are those that are repaired, or reconditioned, and made available for service, instead of being disposed off.

Figure 10.1 shows classification of maintenance items. In the following sections, we discuss inventory models and methods of managing those.

Fig. 10.1 Classification of maintenance items



10.3 Managing Fast-Moving Items

In the previous chapters, we reviewed inventory models to compute the order quantity and reorder levels for items, as well as the timing of the review. Fast-moving items can be managed using any of the deterministic (Chap. 3) or stochastic models (Chap. 6) we discussed earlier in this book. Therefore, only an outline of this philosophy will be presented in this section.

The primary function of the maintenance stores is to minimize the total inventory costs of maintenance inventories by balancing the holding costs and the shortage costs incurred due to equipment downtime.

When the demand for a maintenance item is steady, then the EOQ formula (Chap. 3), shown below, can be used to determine the order size:

$$Q = \sqrt{\frac{2DC_o}{iC}} \tag{10.1}$$

As discussed earlier, demand for a maintenance item is caused by

- the need to carry out preventive and routine maintenance actions, such as those recommended by the equipment manufacturer, and issues found during regular cleaning of equipment;
- the need to carry out corrective maintenance actions, such as fixing a failed equipment.

In certain situations, the demand for maintenance items can be forecast, but most of the demand is uncertain. We, therefore, need to use our learnings from Chap. 6 to estimate the reorder level. We know that the reorder level can be determined by

$$s = \bar{d}L + z\sigma_i\sqrt{L} \tag{10.2}$$

where

- $\bar{d}$ is the average lead time demand;
- L is the lead time;
- z is the standardized normal variate (or the number of standard deviations for a specified service level;
- σ_i is standard deviation of the *lead time demand.*

It is important to ensure that the units for mean, standard deviation, and lead time are the same. For example, the mean, standard deviation, and lead times must all be in days (or months, or years).

Solved Problem 10.1
Sam Motors is an authorized service station for passenger cars. The demand for engine oil seals at Sam Motors is 100 per month with a standard deviation of 5 per month. The lead time for procuring these seals from the market is 1 month. Each seal costs $1 and the ordering cost is $5 per order. If the inventory carrying rate used

is 10% per annum, compute the economic order quantity and the reorder point if a cycle service level of 85% is desired. Assume demand during lead time to be normally distributed.

Solution

- Mean monthly demand, $\bar{d}$, is 100 units (or 1200 units per year).
- Standard deviation of demand, σ_i, is 5 units.
- Lead time is 1 month.
- Ordering cost, C_o, is \$5 per order.
- Cost of item, C, is \$1.
- Desired service level, z, is 0.85.
- Carrying interest rate, i, is 0.10.

Using Eq. 10.1, we can determine the optimal order quantity. Substituting the values, we get

$$Q = \sqrt{\frac{2 \times 1200 \times 5}{0.10 \times 1}} = 346 \text{ units}$$

The standard deviation of the lead time demand, σ_d, is 5 units. The problem states that a service level of 85% is desired. We can use the NORM . S . INV(p) function in MS Excel to obtain the standard normal variate for a specified service level, which in this case is 0.85. We get

$$z = \text{NORM.S.INV}(0.85) = 1.04$$

We can now use Eq. 10.2 to determine the reorder level. Substituting the values of d , L , z, and σ in Eq. 10.2, we get

$$s = \bar{d}L + z\sigma_d = (100 \times 1) + (1.04 \times 5) = 105 \text{ units}$$

The ordering policy for the given problem is as follows:

Place an order for 346 units when the inventory level reaches 105 units.

10.4 Managing Slow-Moving Items

Mitchell (1962) has categorized slow-moving items[1] into

- special items;
- items that provide adequate warning; and
- true standby spares (or items that provide inadequate warning).

Management of the above categories of maintenance items is the focus of this section.

[1]Primarily for coal industry, but can be applied to others as well.

10.4.1 Special Items

A special item, or a special spare, is one that is bought for use or required for a planned maintenance activity on a specific date (See Kelly 2006; Mitchell 1962; Vrat 2014). Since the date for consumption (or usage) of this part is known well in advance, it can be ordered in such a way that it arrives just in time. No inventory of this type of spare is required to be maintained. Since the order size is usually one (e.g., one part or one kit or one set), inventory managers need to worry only about the timing of order placement, taking into consideration the procurement lead time.

Procurement lead times may vary, and it is a usual practice to add a buffer time to the procurement lead time. This is called the *safety lead time*. In the past, researchers have used normal distribution to model procurement lead times. Figure 10.2 shows a normal distribution with probability values in terms of mean and standard deviations. Note that 68% of the area under the curve falls between −1 and +1 standard deviation, 95% of the area falls between −2 and +2 standard deviations, and 99% of the area under the normal curve falls between −3 and +3 standard deviations. Let us now discuss how we can use this to manage the timing of procurement orders for specials. Consider a special spare that has a standard lead time of 45 days with a standard deviation (SD) of 5 days. The following may be kept in mind before placing an order:

- There is a 68% chance that the ordered special item would arrive anywhere between the 40th day (mean – one SD) and 50th day (mean + one SD).
- There is a 95% chance that the ordered special item would arrive anywhere between the 35th day (mean – two times SD) and 55th day (mean + two times SD).
- There is a 99% chance that the ordered special item would arrive anywhere between the 30th day (mean – three times SD) and 60th day (mean + three times SD).

If the inventory manager wants a high level of assurance (say, 99%) of receiving the special, she would want to use a safety lead time of three times the standard deviation, or 15 days. She would need to place an order 60 days in advance

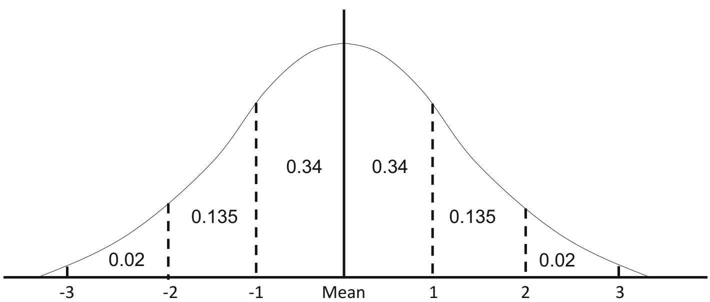


Fig. 10.2 Normal distribution

(45 days' standard lead time + 15 days' safety lead time) to assure that 99 out of 100 times the special would be received on time to perform the planned maintenance activity.

10.4.2 Items that Provide Adequate Warning

Some parts provide sufficient indication of an impending failure. Warnings such as noise, excessive vibration, abnormal temperature, etc. are indications that items are likely to fail soon. Since adequate warning is provided, the inventory policy for this type of spare should be to *not* stock this item, and place an *expedited* order the moment an indication of an impending failure is observed.

10.4.3 Items that Provide Inadequate Warning

In this section, we discuss analytical as well as graphical methods of managing inventory of items that provide little or no warning of an impending failure.

10.4.3.1 Analytical Method

Mitchell (1962) has shown that the average annual cost of holding maximum stock of N spares in inventory is given by

$$C_N = C_h\left[N - \frac{L}{T}\left\{\frac{\sum_{n=0}^{N-1} p(n)}{\sum_{n=0}^{N} p(n)}\right\}\right] + \frac{C_s}{T}\left\{\frac{p(N)}{\sum_{n=0}^{N} p(n)}\right\} + \frac{C_o}{T}\left\{\frac{1-p(N)}{\sum_{n=0}^{N} p(n)}\right\} \tag{10.3}$$

where

- C_h is the cost of holding a slow-moving spare in inventory
- C_s is the cost of stockout (losses due to not holding a slow-mover when needed)
- C_o is the ordering cost per order
- L is the average lead time
- T is the time between failures (demand)
- $p(n)$ is the probability of n slow-moving parts demanded during lead time. Demand is assumed to follow a Poisson distribution represented by Eq. 10.4
- N is the maximum number of spares

$$p(n) = \left(\frac{L}{T}\right)^n \times e^{-\frac{L}{T}} \times (n!)^{-1} \tag{10.4}$$

The objective function Eq. 10.3 is to determine N such that C_N is minimal. The value of N is unlikely to exceed 3 units. See Vrat (2014). In some cases, the term involving ordering cost is not included in calculation, because it is relatively very small compared to the holding and shortage costs.

Solved Problem 10.3

Demand for a grinding ring of a coal-pulverizing mill is 1 per year. The lead time to procure this item is 1 year. If the shortage cost is \$250,000 per unit and the holding cost is \$50,000 per unit, compute the number of spare grinding rings that need to be kept in stock. Assume lead time follows a Poisson distribution.

Solution

In this problem, we have been given the following:

- C_h, the cost of holding a spare in inventory is \$50,000 per unit.
- C_s, the cost of stockout (or shortage) is \$250,000 per unit.
- L, the average lead time is 1 year.
- T, the time between failures is 1 year.
- $p(n)$ follow a Poisson distribution; we can use the Poisson distribution function in MS Excel.

$$\text{POISSON.DIST}(n, \text{mean}, \text{Flag})$$

where

- n takes different values (0, 1, 2 etc.);
- Mean is $\left(\frac{L}{T}\right)$, which is 1.

Flag parameter can be set to False when we need just the individual Poisson probability, and True when we need the cumulative value. Table 10.1 shows the results of substituting the above values in Eq. 10.3 for different values of N (0, 1, 2, etc.) (Fig. 10.3).

It is noticed that the total costs corresponding to $N = 2$ is minimal. Therefore, it is best to maintain an initial inventory of two grinding rings.

Table 10.1 Summary of TIC for different values of N

N	Holding costs	Shortage costs	Total costs
0	0	250,000	250,000
1	250,000	125,000	150,000
2	60,000	50,000	110,000
3	103,125	15,625	118,750
4	150,769	3846	154,615

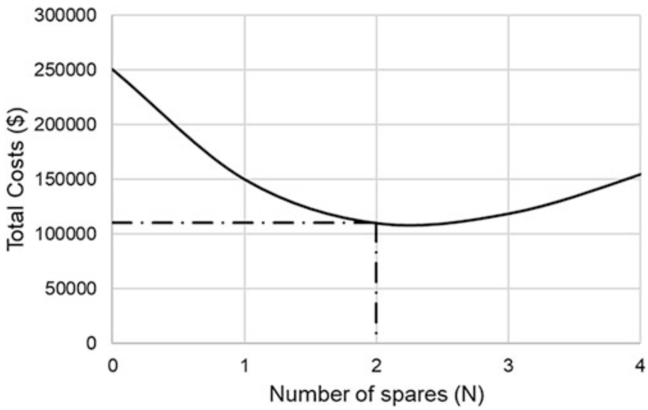


Fig. 10.3 Total cost for different values of N (Initial spares)

10.4.3.2 Graphical Method

Mitchell (1962) has also developed indifference curves as shown in Fig. 10.4. An indifference curve is one where all the points lying on the curve have the same utility value. For example, all the points lying on the line O–H in Fig. 10.4 have the same utility value (in this case, have the same cost, C_N). Similarly, all the points lying on $C_1 = C_2$(represented by other diagonal lines in Fig. 10.4) have the same costs. It is noticed that $C_1 = C_2$ is a function of lead time, and hence there would be one curve for each value of lead time, for $C_1 = C_2$. Figure 10.4 shows indifference curves for various values of lead times ($L = 1, 2, 3, 4, 5, 6, 12,$ and 18 months). When the values of lead time, demand, cost of holding, and stockout are known, the following decision rules may be used. The ratio of stockout to holding costs[2], ${}^{C_s}/_{C_h}$, and the demand for the spare, D, correspond to a point in the graph. Depending on where this point lies on the graph, Table 10.2 presents decision rules that would help determine the number of spares that need to be held initially in the inventory. The order size is always 1.

Solved Problem 10.4
Consider the data provided in Solved Problem 10.3. Use the graphical method to determine the initial number of spares that need to be maintained in the stores.

Solution
We have the following data with us:

- C_h, the cost of holding a spare in inventory is \$50,000 per unit.
- C_s, the cost of stockout (or shortage) is \$250,000 per unit.
- L, the average lead time is 1 year (12 months).

[2]Called the run-out cost in Mitchell (1962).

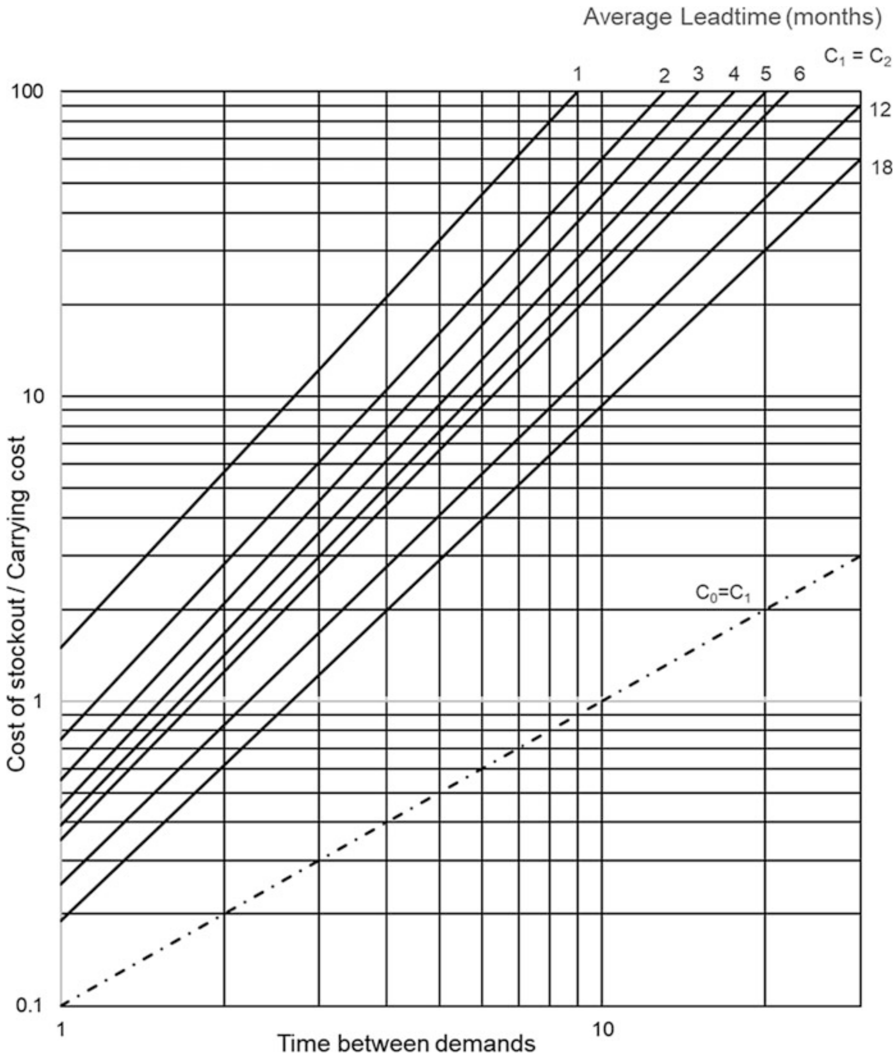


Fig. 10.4 Indifference curves for slow-moving spares (Reproduced from Mitchell (1962), with permission. See also Kelly (2006), pp. 146)

Table 10.2 Decision rules for slow-moving spares – Graphical method

Point of graph is	Decision
below $C_0 = C_1$ (O-H in Fig. 10.4)	Do not hold any spare
between $C_0 = C_1$ and $C_1 = C_2$	Hold one spare
Above $C_1 = C_2$	Hold two spare

- T, the time between failures is 1 year.
- The ratio of the stockout cost to holding cost is 5.

From Fig. 10.4, we see that the coordinates of this point (1,5) lie above $C_1 = C_2$ (that corresponds to $L = 12$ months). From decision rules presented in Table 10.2, we see it is best to hold two spares in the inventory.

Solved Problem 10.5
Determine the initial number of spares that need to be maintained in the stores if the failure rate is 0.2 per year, the lead time is 2 years, and stockout cost to holding cost ratio is 0.3.

Solution
Since the demand is 0.2 per year, the time between issues is 1/0.2 = 5 years.
The stockout cost to holding cost ratio is 0.3.
From Fig. 10.4, we see that the coordinates of this point (5,0.3) lie below $C_0 = C_1$. From decision rules presented in Table 10.2, it is best to hold no spares in the inventory.

10.5 Managing Rotables

Rotables is a special category of maintenance item that is not consumed during a maintenance operation but is reconditioned and returned to service. Items such as this are also referred to as recoverable or repairable spare. Consider an example of a naval ship that returns to the shore to undergo repairs. A failed part on the ship, say, the engine, is removed and is replaced with an engine in working condition from the stores (inventory). The failed engine is not disposed off, but is sent to the repair shop for reconditioning. After reconditioning, the engine, which is now in a serviceable state, is sent to the stores where it awaits deployment on ships as a need arises.

Inventory policy, in this case, would be to determine the *initial* number of rotables that would need to be maintained to satisfy the desired service level. The most common way of treating this problem is to use the queuing model where failures of equipment are treated as customers, and rotables are considered to be the queuing channels. A stockout situation occurs when a customer arrives but all rotables are busy, that is, no rotables are available in stores. Doeh (1960) has shown that the probability of stockout, $p(0)$, can be given by

$$p(0) = \frac{\rho^n / n!}{\sum_{j=0}^{n} \rho^n / n!} \tag{10.5}$$

We use the same symbols used in any standard queuing theory text. In Eq. 10.5, ρ is the utilization factor or the traffic intensity, and n is the number of rotables.

ρ can be determined using the ratio of the mean arrival rate to the mean service rate. In other words,

$$\rho = \frac{\lambda}{\mu} \tag{10.6}$$

where λ is the mean inter-arrival rate and μ is the service rate. Eq. 10.5 can be used to determine the optimal number of rotable spares for a desired service level. The model assumes arrivals follow a Poisson distribution.

Solve Problem 10.6

Historical data show that the oil pump of the pulverizer has failed 200 times over 5 years. It takes 15 days to repair an oil pump and return it to a serviceable state. Determine the number of spare oil pumps that need to be stocked to obtain a service level of 95%.

Solution

In this problem, the failure rate, λ

$$= \frac{200}{5} = 40 \text{ failures per year}$$

The mean repair rate is 15 days or

$$\frac{1}{\mu} = \frac{15}{365}$$

Therefore,

$$\rho = \frac{\lambda}{\mu} = \frac{40 \times 15}{365} = 1.65$$

Table 10.3 shows the results for different values of n. The number of rotables that need to be stocked is between four and five to obtain a service level of 95%.

An alternative technique that has been used for the determination of a rotable inventory policy has been described by Hodges (as quoted by Kelly 2006). It is assumed that a rotable has a failure rate of p per year. If there are N identical equipment using these rotables, the combined rotable failure rate is Np. If the mean time to repair is T years, then the savings in downtime D if one rotable is held in inventory is

Table 10.3 Optimal number of rotables for a given traffic density

n	0	1	2	3	4	5
$\rho^n/_{n!}$	1	1.65	1.36	0.75	0.31	0.10
$\sum_{j=0}^{n} \rho^n/_{n!}$	1	2.65	4.01	4.76	5.07	5.17
$P(0)$	1.00	0.62	0.34	0.16	0.06	0.02

$$D = 1 - e^{-NpT} \text{ unit years per year} \tag{10.7}$$

If two rotables are held, then the savings in downtime is

$$D = 1 - (1 + NpT)e^{-NpT} \text{ unit years per year} \tag{10.8}$$

Similarly, if x rotables are held in stores, the downtime savings in unit years per year is given by

$$D = 1 - e^{-NpT}\left[1 - NpT + \frac{1}{2}NpT^2 + \frac{1}{(x-1)!}NpT^{(x-1)}\right] \tag{10.9}$$

Figure 10.5 shows the relationship between the savings in downtime D and NpT for different values ($x = 1$ and 2) of rotables.

Let us now apply this to Rosetta's case. Rosetta's uses a large automated tortilla-making machine that includes a dough loading section, a pressing section, a spiral heating (carousel) section, and a packaging section. The spiral heating section is driven by a powerful three-phase, 230 V electric motor, that is used 365 days a year for 16 hours each day. In the event of a motor failure, production losses incurred by Rosetta's would be \$100 per hour. The failure rate (p) of the motor is 2 per year, and the mean time to repair (T) is 3 months. Rosetta's has four such identical tortilla-making machines, and currently has one spare motor (rotable) in their stores. In this case,

$$NpT = 4 \times 2 \times \frac{3}{12} = 2$$

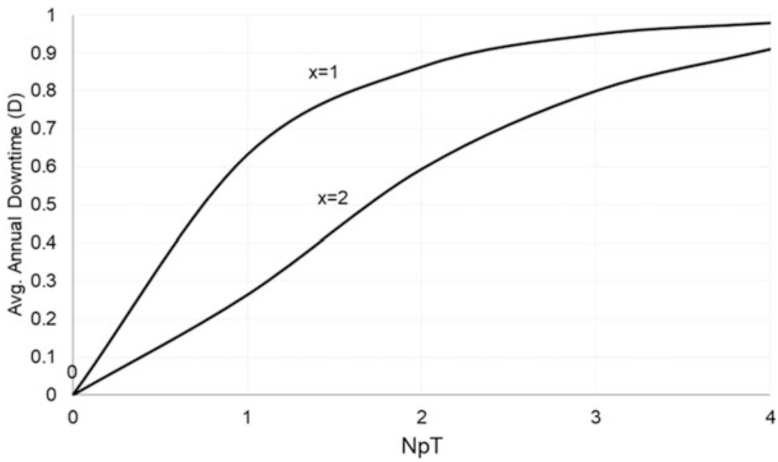


Fig. 10.5 Relationship between NpT and equipment downtime

Since Rosetta's has one spare rotable, we can use Fig. 10.5 to estimate the average annual downtime of equipment for $NpT = 2$ and $x = 1$, which is 0.85. The annual cost of downtime is

$$= 365 \times 24 \times 100 \times 0.85 = \$744,600$$

If Rosetta's employs one more spare, then the estimated annual average downtime would be (from Fig. 10.5) 0.63.

$$= 365 \times 24 \times 100 \times 0.63 = \$551,880$$

The savings would be $\$744,600 - \$551,880$= \$192,720 per year.

Solved Problem 10.7
Use Hodges method to compute the annual downtime cost for the following scenario: $N = 2$, $p = 0.5$, $T = 1$, and $x = 1$. Assume 365×24 work hours and production losses of \$50 per hour.

Solution
In this case,

$$NpT = 2 \times 0.5 \times 1 = 1$$

From Fig. 10.5, the average annual downtime, D, for $NpT = 1$ and $x = 1$ is 0.63. The total downtime cost is $365 \times 24 \times 50 \times 0.63 = \$276{,}000$

10.6 Summary

In this chapter, we discussed management of a different class of items – maintenance items. We discussed management of fast-moving items, slow-moving items, and rotable spares. Inventory managers can use any of the deterministic or stochastic techniques presented in the earlier chapters to manage fast-moving spares. However, the same techniques may not be applicable to manage slow-moving spares because the consumption of spare parts is random, and reliability of historical records is low. Even though the demand is low, organizations would still need to maintain an inventory of these parts because they may not be available a few years later as original equipment manufacturers would no longer be producing those parts. In our discussions in this chapter, we presented both graphical and analytical methods that can help inventory managers determine the initial number of slow-moving spares they would need to keep in their stores. Finally, in this chapter, we discussed a mathematical model to manage rotables or repairable items that are reconditioned and made available for service.

10.7 Case Study: Managing Spare Parts at a Thermal Power Plant

A power generation unit of a thermal power plant consists of several subsystems – the coal washing section, the desalination system, the fuel system, the steam generation system, the economizer section, and the cooling section. Relatively, the fuel system encounters a higher number of failures than the other subsystems (Shenoy and Bhadury 1998).

At a very high level, the fuel system of a thermal power plant consists of a coal crushing unit, or a pulverizer, a set of coal-carrying ducts and pipes and a set of burners. The function of a pulverizer is to grind (crush) coal into fine particles. Raw cleansed coal is transported via open coal-carrying ducts into the pulverizer. The pulverizer is a large processing chamber that has grinding elements consisting of grinding balls that roll between upper and lower grinding rings. Raw coal gets entangled between the grinding elements and is ground to fine particles. Air at high pressure is sent into the grinding chamber using Primary Air Fans. The lighter, ground coal particles are blown through the closed coal-carrying pipes to the burner section, while the larger, coarser coal particles remain inside the chamber till ground further. The coal-carrying chutes are large pipes (12 in diameter) that connect the pulverizer section to the burners. The fine ground coal particles brush against the sides of the chutes as it is transported. The abrasive wear is more prominent at the bends of these chutes than in straights. A frequently occurring failure is leakage of ground coal from these bends that require replacement of these bend sections.

A pulverizer is a high-maintenance equipment that needs to be overhauled once every 18 months. Overhaul involves inspection and replacement of all worn out parts. Table 10.4 describes the key components of the fuel system.

Table 10.4 Case study – Maintenance materials

	Components/parts	Failure rate	How does failure occur
A	Grinding elements (ball and race)	random	Grinding elements – especially grinding balls break randomly, more so when undetected foreign elements (such as iron) get into the grinding chamber
B	Sealing and primary air fan	Low	Air fans do not fail but the motors driving them do fail
C	Stirrup assembly	High	High rate of failure, bolt breaks due to high tension. Frequent replacement required
D	Pulverizer main shaft	Very low	Very rare that this fails, but possibility exists. If this happens, the entire pulverizer will need to be dismantled and shaft replaced
E	Oil pump with drive gear	High	Failure due to clogging of oil pump, contamination
F	Burner set	High	Failure occurs due to clogging of coal particles in burners. More frequent during winter
G	Coal-carrying chutes (at bends)	High	Abrasion/wear-out caused due to rubbing of coal against the sides

Case Study Questions

(a) Using concepts learned in this chapter, categorize the components into fast-moving, slow-moving (adequate warning, inadequate warning, and specials), and rotables.
(b) For each of the parts, state the inventory policy you would use to manage.

10.8 Practice Problems

Problem 10.1

An engine has a mean failure rate of 1 per week and a repair time of 2 weeks. Use Doeh's method to determine the number of spare engines that need to be stocked to obtain a service level of 95%.

Answer

In this problem, $\lambda = 1$ per week and $\frac{1}{\mu} = 2$ weeks. Therefore, $\sigma = \frac{\lambda}{\mu} = 2$. Using Eq. 10.5, we can compute the probability of acceptable stockout shown in Table 10.5.

The number of spare engines to be stored to obtain a 95% service level is five engines.

Problem 10.2

What changes would you make to your stocking policy in Problem 10. 1 if the service time were reduced to 1 week.

Answer

For $\sigma = 1$, the probability of acceptable stockout computed using Eq. 10.5 is shown in Table 10.6. The number of spares required to achieve a service level of 95% would be four.

Problem 10.3

Demand for a high-investment slow-moving spare is 1 per year. The lead time to procure this item is 4 months. If the shortage cost is $100,000 per unit and the

Table 10.5 Optimal number of rotables for $\sigma = 2$

n	0	1	2	3	4	5
$\rho^n/_{n!}$	1	2	2	1.333	0.667	0.267
$\sum_{j=0}^{n} \rho^n/_{n!}$	1	3	5	6.333	7	7.267
P(0)	1	0.667	0.400	0.211	0.095	0.037

Table 10.6 Optimal number of rotables for $\sigma = 1$

n	0	1	2	3	4	5
$\frac{\rho^n}{n!}$	1	1	0.5	0.167	0.667	0.042
$\sum_{j=0}^{n} \rho^n/_{n!}$	1	2	2.5	2.67	2.71	2.72
P(0)	1	0.5	0.2	0.063	0.015	0.003

holding cost is $25,000 per unit, compute the number of spare grinding rings that need to be kept in stock. Use Mitchell's graphical method.

Answer
The ratio of shortage to holding cost is 4. The lead time is 6 months. Looking up Fig. 10.4, we see the coordinate is above the diagonal line for a lead time of 6 months. So, the decision would be to stock two spares.

Problem 10.4
Demand for spares is one per year. The ratio of stockout to holding cost is 1 while lead time is 2 months.

(a) Determine the number of spares that need to be maintained.
(b) If the lead time is reduced to 1 month, how would this impact the stocking policy?

Answer
The ratio of shortage to holding cost is 1. The lead time is 2 months. Looking up Fig. 10.4, we see the coordinate is above the diagonal line for a lead time of 2 months. So, the decision would be to stock two spares. If lead time is reduced to 1 month, the number of spares to be held is one.

Problem 10.5
Use Hodges method to compute the annual downtime cost for the following scenario: $N = 3, p = 1, T = 1$, and $x = 2$. Assume 365×24 work hours and production losses of $50 per hour.

Answer
$NpT = 3, x = 2$. Looking up Fig. 10.5, the average downtime, D, for $NpT = 3$ and $x = 2$ is 0.8. The total downtime cost is $365 \times 24 \times 50 \times 0.8 = \$350{,}400$.

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Chapter 11
Multi-echelon Inventory Models

11.1 Multi-echelon: Definition

To increase their market share, Rosetta's has now decided to allow a sole franchisee in León to sell the Rosetta brand of tortillas. The franchisee places an order for a certain number of packs of tortillas. This costs them $4 per order. Rosetta's supplies each pack of tortillas to the franchisee at a price of $10. The franchisee can retail a pack of tortillas at $22. Inventory carrying rate is 30% per annum and the annual demand for tortillas is 6000 packs. How do we determine the ordering quantities for the franchisee as well as Rosetta's? Rosetta's "incurs" an order cost of $8 per order when they procure the packets from their kitchen. Also, we assume the tortillas have an indefinite shelf life.

In this chapter, we address a multi-echelon (simple, two-stage) inventory decision-making problem.

Inventory models discussed in the previous chapters dealt with items stocked in one location. In reality, most systems involve multiple locations or stages. For example, customers arrive at a retail shop to buy items they need. The stock of items at the retail shop gets depleted as demand arises. At some point, the retail shop owner places a replenishment order with a warehouse, which, when received, increases the quantity of items on hand at the retail shop. Similarly, as the inventory level in the warehouse gets depleted the warehouse manager needs to place an order with the manufacturer to supply fresh stock, and so on. Each of these entities

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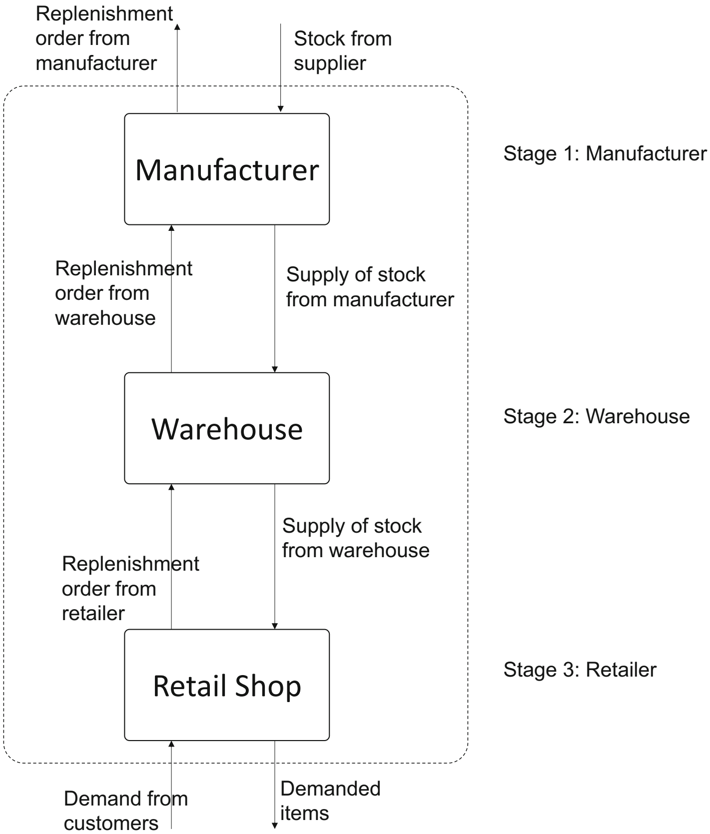

Fig. 11.1 Multi-echelon System

involved in the discussion – the retailer, the warehouse, and the manufacturer – are referred to as stages or echelons. Figure 11.1 shows a schematic of a multi-echelon inventory system.[1]

11.2 Two-Stage Inventory Model: Deterministic Demand

Consider a simple serial system consisting of just two stages – one warehouse and one retail outlet – dealing with a single item. The retail outlet buys its supplies from the warehouse to meet the demand from customers, and the warehouse sources its

[1]Multistage manufacturing systems also use the same concept. However, in this book we have restricted our discussions to retail–warehouse distribution inventory systems.

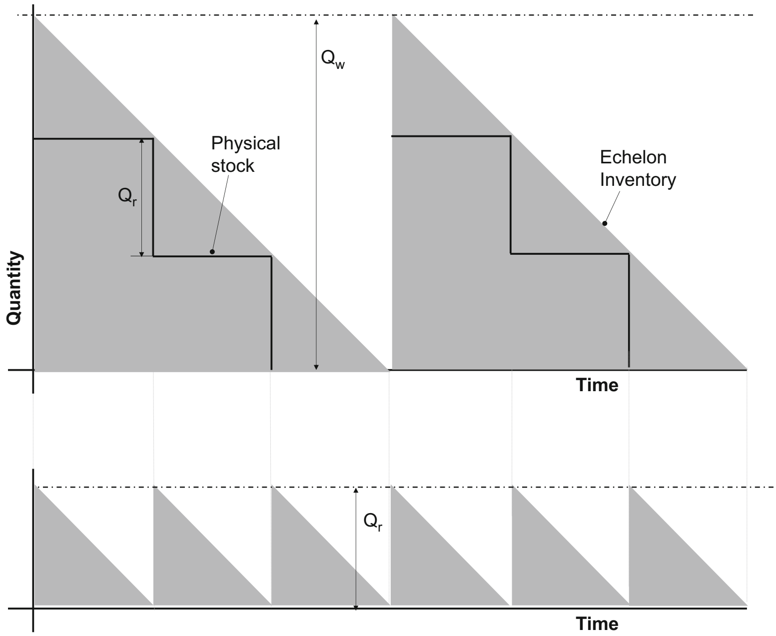

Fig. 11.2 Two-stage inventory system

supplies from an external source. Figure 11.2 shows a schematic of the two-stage inventory system.

Let us further assume that the demand for items is constant (deterministic) at D units per year. Other notations used in this model are as follows:

- C_{ow} is the ordering cost per order for the warehouse.
- C_{or} is the ordering cost per order for the retailer.
- C_w is the unit price of the item at the warehouse.
- C_r is the unit price of the item at the retailer.
- Q_w is the order quantity for the warehouse.
- Q_r is the order quantity for the retailer.
- i is the carrying rate per annum.

The average inventory at the retail outlet is

$$\frac{1}{2}Q_r \tag{11.1}$$

The average inventory at the warehouse, however, cannot be determined easily since it does not follow the sawtooth pattern. Clark and Scarf (1960) introduced the concept of echelon inventory and proved that echelon inventory[2] at any given stage

[2]We represent echelon inventory by the symbol I^i.

is equal to the inventory held at that stage plus the inventory held at all downstream stages. In other words, echelon inventory at the warehouse, $I_w^{Ç}$, can be written as

$$I_w^{Ç} = I_w + I_r \tag{11.2}$$

We know that the total inventory cost (TIC) is equal to the sum of the ordering costs and the carrying costs. The ordering cost for the warehouse is

$$\frac{C_{ow}D}{Q_w} \tag{11.3}$$

and the carrying cost for the warehouse is

$$I_w^{Ç} i C_w^{Ç} \tag{11.4}$$

Similarly, the ordering cost for the retailer is

$$\frac{C_{or}D}{Q_r} \tag{11.5}$$

and the carrying cost for the retailer is

$$I_r^{Ç} i C_r^{Ç} \tag{11.6}$$

It should be noted that $C_r^{Ç}$ is the cost of value addition at the echelon. The value addition at the retailer echelon would be

$$C_r^{Ç} = C_r - C_w \tag{11.7}$$

Therefore, the total inventory costs for the complete two-stage system is

$$\text{TIC} = \frac{C_{ow}D}{Q_w} + I_w^{Ç} i C_w^{Ç} + \frac{C_{or}D}{Q_r} + I_r^{Ç} i C_r^{Ç} \tag{11.8}$$

From Fig. 11.2, we can see that the order quantity for the warehouse is an integer multiple of the retailer order quantity. Therefore,

$$Q_w = nQ_r \tag{11.9}$$

Substituting the above, we get

$$\text{TIC} = \frac{C_{ow}D}{nQ_r} + \frac{nQ_r^{Ç}}{2} i C_w^{Ç} + \frac{C_{or}D}{Q_r} + \frac{Q_r^{Ç}}{2} i C_r^{Ç} \tag{11.10}$$

Simplifying and rearranging Eq.11.10, we get

$$\text{TIC} = \frac{D}{Q_r}\left(C_{or} + \frac{C_{ow}}{n}\right) + \frac{iQ_r^{\text{Ç}}}{2}\left(nC_w^{\text{Ç}} + C_r^{\text{Ç}}\right) \tag{11.11}$$

To obtain the minima, we need to differentiate Eq. 11.11 with respect to Q_r and equate to 0. By doing so, we get

$$\frac{\partial \text{TIC}}{\partial Q_r} = -\frac{D}{{Q_r}^2}\left(C_{or} + \frac{C_{ow}}{n}\right) + \frac{i}{2}\left(nC_w^{\text{Ç}} + C_r^{\text{Ç}}\right) = 0$$

or

$$Q_r = \sqrt{\frac{2\left(C_{or} + \frac{C_{ow}}{n}\right)D}{i\left(nC_w^{\text{Ç}} + C_r^{\text{Ç}}\right)}} \tag{11.12}$$

Substituting the value of Q_r in Eq.11.11, we get

$$\text{TIC} = \sqrt{2\left(C_{or} + \frac{C_{ow}}{n}\right)i\left(nC_w^{\text{Ç}} + C_r^{\text{Ç}}\right)D} \tag{11.13}$$

The objective is to find the *integer* value of n such that TIC is minimal. To determine the value of n, we use a simpler function, $f(n)$, and because D and i cannot be 0, and 2 is constant, we have

$$f(n) = \left(C_{or} + \frac{C_{ow}}{n}\right)\left(nC_w^{\text{Ç}} + C_r^{\text{Ç}}\right) \tag{11.14}$$

where $f(n)$ is a function of TIC. Differentiating with respect to n and equating to 0, we get

$$n = \sqrt{\frac{C_{ow}C_r^{\text{Ç}}}{C_{or}C_w^{\text{Ç}}}} \tag{11.15}$$

An integer value of n can be substituted in Eq. 11.9 to determine the values of Q_r and Q_w. If n is not an integer, we need to use Eq. 11.14 to determine the integer values that surround n. Following rules (Silver et al. 1998) can be used to determine n:

$$n = n_1, \text{ if } f(n_1) \le f(n_2)$$
$$n = n_2, \text{ if } f(n_1) > f(n_2)$$

Let us now solve the ordering quantity problem for Rosetta's running example and their sole franchisee using the multi-echelon concept. We have the following information with us:

- C_{ow} is \$8 per order
- C_{or} is \$4 per order
- C_w is \$10
- C_r is \$22
- i is 0.3

The first step is to compute $C_r^{Ç}$ and $C_w^{Ç}$. From Eq. 11.7, we have

$$C_r^{Ç} = C_r - C_w = 22 - 10 = 12$$

In a two-stage inventory model, $C_w^{Ç} = C_w = 10$.

The next step is to compute n. From Eq. 11.15, we have

$$n = \sqrt{\frac{C_{ow}C_r^{Ç}}{C_{or}C_w^{Ç}}} = \sqrt{\frac{8 \times 12}{4 \times 10}} = 1.55$$

Since n is not an exact integer, we need to find $f(n_1)$ and $f(n_2)$ using integer values of n_1 and n_2 that bind n. Using $n_1 = 1$ and $n_2 = 2$ in Eq. 11.14, we get

$$f(n_1) = \left(C_{or} + \frac{C_{ow}}{n_1}\right)\left(n_1 C_w^{Ç} + C_r^{Ç}\right) = \left(4 + \frac{8}{1}\right)(10 + 12) = 264$$

$$f(n_2) = \left(C_{or} + \frac{C_{ow}}{n_2}\right)\left(n_2 C_w^{Ç} + C_r^{Ç}\right) = \left(4 + \frac{8}{2}\right)(20 + 12) = 256$$

Since $f(n_1) > f(n_2)$, the decision would be to use $n = n_2 = 2$. The final step is to compute the optimal lot size for the franchisee (retailer). Using Eq. 11.12, we get

$$Q_r = \sqrt{\frac{2\left(C_{or} + \frac{C_{ow}}{n}\right)D}{i\left(nC_w^{Ç} + C_r^{Ç}\right)}} = \sqrt{\frac{2 \times \left(4 + \frac{8}{2}\right) \times 6000}{0.3(2 \times 10 + 12)}} = 100$$

From Eq. 11.19, we also have the lot size for the warehouse (Rosetta's), which is

$$Q_w = nQ_r = 2 \times 100 = 200$$

The optimal lot size for the franchisee is 100 packets, while that for Rosetta's to procure from their kitchen is 200 units.

Solved Problem 11.1[3]

Real Great Foods (RGF) is a distributor of frozen food products. They source large quantities of manchego cheese from their local supplier at \$10 per kilogram. They incur an ordering cost of \$5 per order. Fresh manchego cheese, arriving at RGF distribution center, are cut uniformly[4] into thin square slices of 3in × 3 in. Each slice of cut cheese is wrapped in a plastic wrapper. Twenty such wrapped slices are put into a plastic pack that has the RGF logo on it. RGF sells each pack of manchego cheese to retail shops at \$44 per pack of 1 kg. The setup cost for cutting, wrapping, and packaging operation is \$7. If the inventory carrying rate is 20% per annum, compute the ordering quantities. Annual demand is 4275.

Solution

We have the following data:

- $D = 4275$
- $C_{ow} = \$5$
- $C_{or} = \$7$
- $C_w = \$10$ per kilogram
- $C_r = \$44$ per kilogram
- $i = 0.20$

The first step is to compute $C_r^Ç$ and $C_w^Ç$. From Eq. 11.7, we have

$$C_r^Ç = C_r - C_w = 44 - 10 = 34.$$

In a two-stage inventory model, $C_w^Ç = C_w = 10$.

The next step is to compute n. From Eq.11.15, we have

$$n = \sqrt{\frac{5 \times 34}{7 \times 10}} = 1.55$$

Since this is not an integer, we need to find the values of $n_1$and n_2 that surround n. In this case, $n_1 = 1$ and $n_2 = 2$. Substituting the values in Eq. 11.14, we get

$$f(n_1) = \left(7 + \frac{5}{1}\right)[(1 \times 10) + 34] = 528$$

and

$$f(n_2) = \left(7 + \frac{5}{2}\right)[(2 \times 10) + 34] = 513$$

[3]Inspired by numerical illustration in the work by Silver et al. (1998) pp. 481.

[4]In Mexico, this is called Rebanada (in Spanish) or a slice (in English).

Since $f(n_1) > f(n_2)$, we use $n = 2$. Substituting the value obtained in Eq. 11.12, we get

$$Q_r = \sqrt{\frac{2 \times \left(7 + \frac{5}{2}\right) \times 4275}{0.20 \times [(2 \times 10) + 34]}} = 87$$

From Eq. 11.9, we also have the optimal lot size for the warehouse which is

$$Q_w = nQ_r = 2 \times 87 = 174$$

The optimal lot size for the franchisee is 87 kg while that for the warehouse (to procure from their supplier) is 174 kg.

11.3 Two-Stage Inventory Model: Probabilistic Demand

Let us consider a serial system with one warehouse and one retailer. Let us assume that the demand occurs at the retailer and that it is normally distributed with known mean and standard deviation. Following are the notations used in this section:

- C_w and C_r are the unit costs of the item at the warehouse and retailer, respectively.
- R_w and R_r are the reorder points at the warehouse and retailer, respectively.
- Q_w and Q_r are the order quantities at the warehouse and retailer, respectively.
- D_w and D_r are the lead time demands at the warehouse and retailer, respectively.
- σ_w and σ_r are the standard deviation of lead time demand at warehouse and retailer, respectively.
- C_s is the stockout cost expressed in fraction of unit value charged per unit short.
- z is the standard normal variate.

Assuming that $Q_w = nQ_r$, the decision rules[5] used to determine the reorder points at the retailer and warehouse are as follows:

$$R_r = D_r + z\sigma_r \tag{11.16}$$

where z satisfies the following:

$$p(s) \geq \frac{Q_r(C_r - C_w)i}{C_s C_r D} \tag{11.17}$$

and

[5]Silver, Pyke and Petersen, pp. 492–493.

$$R_r = D_{r+w} + z\sigma_{r+w} \tag{11.18}$$

where z satisfies the following:

$$p(S) \geq \frac{Q_r[C_r + (n-1)C_w]i}{C_s C_r D} \tag{11.19}$$

A Note of Multi-echelon Models for Repairable Items

A lot of literature is available on the multi-echelon models for repairable items. Initial work in this area – called METRIC – was done by Sherbrooke (1968). This was based on the research work done by Palm (1938). An improved algorithm on the subject has been presented by Graves (1985). A brief research-oriented essay on the topic can be found in the work by Vrat (2014). These models are very complex and beyond the scope of this book. Interested readers may refer to the specified material for further study.

11.4 Summary

In this chapter, we presented mathematical models to manage inventory items that are maintained at multiple locations. These multi-echelon models are mathematically quite complex. Therefore, only a high-level view of this topic has been discussed in this chapter. We presented a simple two-stage (warehouse and retailer) model with deterministic demand and derived expressions to compute order quantities for both the retailer and warehouse. We also discussed a multi-echelon model for probabilistic demand and computed order quantities assuming demand to be normally distributed.

11.5 Case Study

Hector has been running an ice cream parlor in the Cancun region of Mexico, where demand for ice cream is almost steady throughout the year. He has been able to sell 3500 kg of ice cream each year. For several years he has been a franchisee of Helado Mexicano, a very popular brand of ice cream in Mexico. Hector always used a standard procedure for ordering ice creams. When appropriate, he places an order with the sole distributor of Helado in the region. The distributor uses refrigerated trucks to deliver the required quantity of ice cream the very next day. It cost him $25 to place an order.

After about 15 profitable years, Hector has now decided to expand his business. He has decided he would open another ice cream parlor in upmarket Cancun, about

8 km from his existing parlor. He believes the new parlor would be able to sell as additional 3500 kg of ice cream. He has asked his son Donato to manage that business for him.

Hector and Donato are now deliberating the way inventory in the new business has to be managed. Between them, they are keen they minimize all inventory costs. They have finalized the following arrangement for executing daily operations:

Donato would be running the new ice cream parlor as a franchisee of Hector's business This means he would give his requirement to Hector who would place an order with the local distributor that would meet requirements of both the locations. Trucks would deliver the ice cream at Hector's parlor, which would be broken, and a part of it would be repackaged and sent to Donato's parlor. Hector believes by ordering in bulk he would be able to negotiate a good price since the total order size is expected to be in the range of 7000 kg a year. He thinks he would be able to get ice cream at $125 per kg for this order size. However, in this scheme of things, Donato will also have to bear an "ordering" cost of $50 per shipment since he would have to pick up the ice creams received from Hector's shop. Hector and Donato are quite confident this additional expenditure will be offset by selling an ice cream at a slightly higher price, considering the new shop would be in an upmarket area. A price of $400 per kg is what they think would be a good price to sell in the upmarket area.

Case Study Questions

- Complete the inventory analysis for this case and determine the order quantities for the businesses of Hector and Donato.
- Also, analyze the costs if Donato decides to run his business independently (not as a franchisee).

  Hint: Use the following data (from the case study)
  We have the following data:

- $D = 7000$
- $C_{ow} = \$25$
- $C_{or} = \$50$
- $C_w = \$125$ per kilogram
- $C_r = \$400$ per kilogram
- $i = 0.30$

11.6 Practice Problems

Problem 11.1
The following data pertain to a two-stage serial system Compute the order quantities for the retailer (r) and the warehouse (w).

- $D = 3600$
- $C_{ow} = C_{or} = \$200$

- $C_w = \$125$ per unit
- $C_r = \$300$ per unit
- $i = 0.25$

Answer

$f(n_1) < f(n_2)$, $n = 1$
$Q_w = Q_r = 143$ units

Problem 11.2
The following data pertain to a two-stage serial system. Compute the order quantities for the retailer (r) and the warehouse (w).

- $D = 5000$
- $C_{ow} = C_{or} = \$200$
- $C_w = \$110$ per unit
- $C_r = \$550$ per unit
- $i = 0.25$

Answer

n as an integer = 2. No need to find $f(n_1)$,$f(n_2)$.
$Q_r = 134$ units

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